The problem of optimal endogenous growth with exhaustible resources revisited

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Abstract

We study optimal research and extraction policies in an endogenous growth model in which both production and research require an exhaustible resource. It is shown that optimal growth is not sustainable if the accumulation of knowledge depends on the resource as an input, or if the returns to scale in research are decreasing, or the economy is too small. The model is stated as an infinite-horizon optimal control problem with an integral constraint on the control variables. We consider the main mathematical aspects of the problem, establish an existence theorem and derive an appropriate version of the Pontryagin maximum principle. A complete characterization of the optimal transitional dynamics is given.

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1 Introduction

Endogenous growth theory identifies technological progress as a means of sustaining economic growth despite the reliance on exhaustible resources as inputs to production. The supply of an exhaustible resource may limit growth, unless the economy can either substitute away from the resource or increase the efficiency of the resource's use to offset its scarcity. Can an optimal research and extraction policy compensate the negative effects on production (consumption) that arise due to scarcity of the exhaustible resource?

Existing literature in the tradition of Dasgupta-Heal-Solow-Stiglitz [8, 21] offers an affirmative answer in a scenario in which production requires the resource but the accumulation of knowledge does not. Our point of departure is the 'toy economy' model by Charles Jones [15], as one of the simplest models of endogenous growth. We show that

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resource-dependency may preclude perpetual growth along a welfare-maximizing output trajectory if technical progress depends on the resource, or, as was advocated by Jones in [14, 15], technical progress shows weak scale effects, or the economy is too small. The possibility of the research sector being dependent on an exhaustible resource challenges the feasibility of perpetual growth, while the strong scale condition seems less profound in view of many existing models that obtain balanced positive growth without it, see [18, 24]. The minimum size condition is the least restrictive of all conditions.

We thus show that welfare-maximizing growth can be either perpetual or transient, and derive optimal research and extraction policies in each scenario. Perpetual growth is balanced and the optimal research policy allocates a constant fraction of the labor force to research. Perpetual growth is feasible even in the absence of population growth and is thus fully-endogenous. Perpetual growth becomes unfeasible if technical progress requires the resource or has weak scale effects. In this more realistic scenario it is optimal to pursue a certain ratio of the knowledge to the resource stock. In the resulting 'rise and decline' scenario output grows initially but stagnates and eventually declines following stagnation of the knowledge stock. In either scenario it is optimal to deplete the resource according to the well-known Hotelling rule.

The model is formulated as an infinite-horizon optimal control problem whose solution is a welfare-maximizing dynamic research and extraction policy. The model includes an integral constraint (in L^1-space) associated with a finite stock of an exhaustible resource. Such integral constraints on control (policy) variables are the defining feature of a class of models in the resource and growth literature (see examples in [23]).

Due to the unbounded nature of controls corresponding to the extraction policy, we cannot directly appeal to the standard results on existence of an optimal control in the class of locally bounded measurable functions (such results usually rely on pointwise boundedness conditions; see, for example, [7]). We overcome this difficulty by reducing our problem to one without integral constraints. This allows us to prove an existence result and apply a version of the Pontryagin maximum principle for problems with a dominating discount developed in [3, 4] to fully characterize the optimal transitional dynamics.

The infinite time horizon gives rise to specific mathematical features of the Pontryagin maximum principle [16]. The most characteristic feature is that the standard transversality conditions may fail (see examples in [4, 10, 20]). There exist modifications of the Pontryagin maximum principle that pay attention to this phenomenon [3–6, 19]. Yet our problem fails to satisfy the assumptions imposed in them due to the integral constraint.

When the Hamiltonian is concave, some infinite horizon problems can be solved by means of well-known sufficient conditions [2, 19]. This is a standard way of solving many optimal economic growth problems (see [1]). Nevertheless, even in our simple model the concavity of the Hamiltonian cannot be asserted for all relevant parameter values.

In this paper we follow the more general approach based on necessary conditions and an existence theorem. It should be stressed that without an existence theorem one cannot be sure that a path satisfying the necessary conditions exists, or that one of the paths satisfying the necessary conditions is indeed a solution (see the discussion in [17]).

2 The model

In the following two-sector endogenous growth model the production sector yields output that is consumed, while the research sector augments the productivity of the production means. Both sectors require an exhaustible resource as an input. There are constant returns to scale in production, and either weak or strong scale effects in the research sector.

At every instant $t \in [0, \infty)$, the economy produces output $Y(t) > 0$, which is assumed to be described by a Cobb–Douglas production function:

$$Y(t) = A(t)^{\varkappa}[L - L^A(t)]^{\alpha} R_1(t)^{1-\alpha} \qquad \text{where} \quad \alpha \in (0,1) \quad \text{and} \quad \varkappa > 0. \tag{1}$$

Here $A(t) > 0$ is the current knowledge stock and $R_1(t) > 0$ is the quantity of the exhaustible resource used in production. The population (total labor supply) is fixed at $L > 0$. Part of the labor $L - L^A(t)$ is employed in production, while the other part $L^A(t) \in [0, L)$ is allocated to research.

The amount of new knowledge produced at time t depends on the hitherto accumulated knowledge, the number of researchers and the portion of the exhaustible resource used in research:

$$\dot{A}(t) = A(t)^{\theta}[L^A(t)]^{\eta} R_2(t)^{1-\eta} \qquad \text{where} \quad \eta \in (0,1] \quad \text{and} \quad \theta \in (0,1]. \tag{2}$$

Here $R_2(t) \geq 0$ is the quantity of the exhaustible resource used in research; typically $R_2(t)$ is small compared to $R_1(t)$. The initial knowledge stock is given by $A(0) = A_0 > 0$. If $\theta \in (0,1)$, then growth rate of the knowledge stock decreases while the knowledge stock expands. The case of $\theta < 0$—when the expansion of knowledge is progressively more difficult—has also been considered in the literature (see, e.g., [14]). Empirical evidence supports the idea of weak scale effects, i.e. $\theta < 1$, in the production of knowledge. We retain $\theta = 1$ as a special case of strong scale effects.

The fact that the stock of the exhaustible resource is finite imposes the following integral constraint on the controls $R_1(\cdot)$ and $R_2(\cdot)$:

$$\int_0^{\infty} \big[R_1(t) + R_2(t)\big]\, dt \leq S_0, \tag{3}$$

where $S_0 > 0$ is the initial supply of the exhaustible resource.

The welfare is measured by a discounted logarithmic utility function, maximizing which amounts to maximizing future growth rates. This leads to the following objective functional for the economy (see (1)):

$$\begin{aligned} J(A(\cdot), L^A(\cdot), R_1(\cdot)) &= \int_0^{\infty} e^{-\rho t}\big\{\ln[Y(t)]\big\}\, dt \\ &= \int_0^{\infty} e^{-\rho t}\Big\{\varkappa \ln A(t) + \alpha \ln[L - L^A(t)] + (1-\alpha)\ln R_1(t)\Big\}\, dt, \end{aligned}$$

where $\rho > 0$ is a subjective discount rate.

Given the parameters $\theta \in (0,1]$, $\alpha \in (0,1)$, $\varkappa > 0$, $\eta \in (0,1]$, $\rho > 0$, $L > 0$ and $S_0 > 0$, the optimization problem $J(A(\cdot), L^A(\cdot), R_1(\cdot)) \to \max$, subject to equation (2) and the resource constraint (3), can be formulated as the following infinite-horizon optimal control problem (P):

$$\dot{A}(t) = A(t)^{\theta}[L^A(t)]^{\eta} R_2(t)^{1-\eta}, \tag{4}$$

$$L^A(t) \in [0, L), \qquad R_1(t) > 0, \qquad R_2(t) \geq 0, \qquad \int_0^\infty \big[R_1(t) + R_2(t)\big]\, dt \leq S_0, \tag{5}$$

$$A(0) = A_0 > 0, \tag{6}$$

$$J(A(\cdot), L^A(\cdot), R_1(\cdot)) = \int_0^\infty e^{-\rho t} \Big\{ \varkappa \ln A(t) + \alpha \ln[L - L^A(t)] + (1-\alpha) \ln R_1(t) \Big\}\, dt \to \max. \tag{7}$$

By an *admissible control* $w(\cdot)\colon [0,\infty) \to \mathbb{R}^3$ in problem (P) we mean a triple $w(\cdot) = (L^A(\cdot), R_1(\cdot), R_2(\cdot))$, $t \geq 0$, of (locally) bounded measurable functions $L^A(\cdot)$, $R_1(\cdot)$ and $R_2(\cdot)$ each of which is defined on the infinite half-open time interval $[0,\infty)$ and satisfies the respective constraints in (5).

An *admissible trajectory* $A(\cdot)\colon [0,\tau) \to \mathbb{R}^1$, $\tau > 0$, corresponding to an admissible control $w(\cdot)$ is a (locally) absolutely continuous function $A(\cdot)$ which is a (Carathéodory) solution (see [9]) of the differential equation (4) on some (finite or infinite) time interval $[0,\tau)$, subject to the initial condition (6).

Due to (4) and the integral constraint in (5), for any admissible control $w(\cdot) = (L^A(\cdot), R_1(\cdot), R_2(\cdot))$ the corresponding admissible trajectory $A(\cdot)$ can be extended to the whole infinite interval $[0,\infty)$. Consequently, in what follows, without loss of generality, we always assume that any admissible trajectory $A(\cdot)$ is defined on $[0,\infty)$.

A pair $(A(\cdot), w(\cdot))$, where $w(\cdot)$ is an admissible control and $A(\cdot)$ is the corresponding admissible trajectory, is called an *admissible pair* (or a *process*) in problem (P).

For any admissible pair $(A(\cdot), w(\cdot))$ the improper integral in (7) converges either to $-\infty$ or to a finite real. Moreover, it is uniformly bounded from above; i.e., there is a number $M \geq 0$ such that

$$\sup_{(A(\cdot),w(\cdot))} \int_0^\infty e^{-\rho t} \Big\{ \varkappa \ln A(t) + \alpha \ln[L - L^A(t)] + (1-\alpha) \ln R_1(t) \Big\}\, dt \leq M, \tag{8}$$

where the supremum is taken over all admissible pairs $(A(\cdot), w(\cdot))$.

Indeed, due to the integral constraint in (5), for any admissible control $w(\cdot)$ we have

$$\int_0^\infty e^{-\rho t} \ln R_1(t)\, dt < \int_0^\infty e^{-\rho t} R_1(t)\, dt < S_0. \tag{9}$$

Further, for an arbitrary admissible trajectory $A(\cdot)$ we have

$$A(t)^{\theta} \leq A(t) + 1, \qquad t \geq 0.$$

Then, due to (4), we obtain

$$\frac{d}{dt} \ln(A(t) + 1) = \frac{\dot{A}(t)}{A(t) + 1} \leq L^{\eta} R_2(t)^{1-\eta}, \qquad t \geq 0,$$

and hence

$$\ln(A(t)+1) \le \ln(A_0+1) + L^\eta \int_0^t R_2(s)^{1-\eta}\, ds \le \ln(A_0+1) + L^\eta \int_0^t (1+R_2(s))\, ds$$
$$< \ln(A_0+1) + L^\eta(t+S_0), \qquad t \ge 0. \tag{10}$$

This inequality immediately implies the following inequality for an arbitrary admissible trajectory $A(\cdot)$:

$$\int_0^\infty e^{-\rho t} \ln A(t)\, dt < \int_0^\infty e^{-\rho t} \ln(A(t)+1)\, dt < \frac{\ln(A_0+1) + L^\eta S_0}{\rho} + \frac{L^\eta}{\rho^2}. \tag{11}$$

Since $L^A(t) \in [0, L)$, $t \ge 0$ (see (5)), inequalities (9) and (11) provide the following uniform estimate for all control processes $(A(\cdot), w(\cdot))$:

$$\int_0^\infty e^{-\rho t} \Big\{ \varkappa \ln A(t) + \alpha \ln[L - L^A(t)] + (1-\alpha) \ln R_1(t) \Big\}\, dt$$
$$< \varkappa \frac{\ln(A_0+1) + L^\eta S_0}{\rho} + \frac{\varkappa L^\eta}{\rho^2} + \frac{\alpha \ln L}{\rho} + (1-\alpha) S_0.$$

This furnishes the proof of inequality (8).

The uniform bound (8) allows us to define an *optimal control* $w_*(\cdot)\colon [0,\infty) \to \mathbb{R}^3$ in problem (P) as a welfare-maximizing triple $w_*(\cdot) = (L_*^A(\cdot), R_{1*}(\cdot), R_{2*}(\cdot))$ of dynamic labor and extraction policies adopted in the research and production sectors. The corresponding trajectory $A_*(\cdot)$ is an *optimal admissible trajectory*.

3 Reduction to a one-dimensional problem without integral constraints

Let us introduce a new state variable $x(\cdot)\colon [0,\infty) \to \mathbb{R}^1$ and new control variables $u(\cdot)\colon [0,\infty) \to (0,\infty)$ and $v(\cdot)\colon [0,\infty) \to [0,\infty)$ as follows:

$$x(t) = \frac{S(t)^{1-\eta}}{A(t)^{1-\theta}}, \qquad u(t) = \frac{R_1(t)}{S(t)}, \qquad v(t) = \frac{R_2(t)}{S(t)}, \qquad t > 0. \tag{12}$$

Here the state variable $S(\cdot)$ represents the current supply of the exhaustible resource. This variable is a (Carathéodory) solution to the following Cauchy problem (for given admissible controls $R_1(\cdot)$ and $R_2(\cdot)$) on $[0,\infty)$:

$$\dot{S}(t) = -R_1(t) - R_2(t), \qquad S(0) = S_0. \tag{13}$$

Note that the case $\eta = \theta = 1$ is not excluded, although in this case the new variable $x(\cdot)$ degenerates into a constant. This case can easily be analyzed directly, but we include it in our general scheme to save the space. Below we show that for $\eta = \theta = 1$ the problem reduces to a zero-dimensional problem, i.e. to a problem in which the utility function depends only on the controls and does not depend on the state variables (hence the control variables take constant values maximizing the utility function at each moment in time).

Note also that $S(t) > 0$ for all $t > 0$, so the quantities $u(t)$ and $v(t)$ are well defined for all $t > 0$. Indeed, if $S(\tau) = 0$ for some $\tau > 0$, then $S(t) = 0$ for all $t > \tau$ and hence $R_1(t) = R_2(t) = 0$ for $t > \tau$, which is precluded by (5). Moreover, $u(\cdot)$ and $v(\cdot)$ are locally bounded measurable functions since $R_i(\cdot)$, $i = 1, 2$, is locally bounded and measurable and $S(\cdot)$ is positive and continuous.

Since $x(\cdot)$ is a (locally) absolutely continuous function, we can calculate its derivative a.e. on $[0, \infty)$:

$$\begin{aligned}
\dot{x}(t) &= (1-\eta)\frac{\dot{S}(t)}{A(t)^{1-\theta}S(t)^{\eta}} - (1-\theta)\frac{\dot{A}(t)S(t)^{1-\eta}}{A(t)^{2-\theta}} \\
&= -(1-\eta)[u(t)+v(t)]x(t) - (1-\theta)\frac{A(t)^{\theta}[L^A(t)]^{\eta}R_2(t)^{1-\eta}S(t)^{1-\eta}}{A(t)^{2-\theta}} \\
&= -(1-\eta)[u(t)+v(t)]x(t) - (1-\theta)[L^A(t)]^{\eta}v(t)^{1-\eta}x(t)^2.
\end{aligned}$$

Thus, $x(\cdot)$ is a Carathéodory solution of the differential equation

$$\dot{x}(t) = -(1-\eta)[u(t)+v(t)]x(t) - (1-\theta)[L^A(t)]^{\eta}v(t)^{1-\eta}x(t)^2, \qquad t > 0, \tag{14}$$

satisfying the initial condition

$$x(0) = x_0 = \frac{S_0^{1-\eta}}{A_0^{1-\theta}}. \tag{15}$$

Now we express the functional $J(A(\cdot), L^A(\cdot), R_1(\cdot))$ (see (7)) in terms of the new variables $x(\cdot)$, $u(\cdot)$ and $v(\cdot)$. Consider the first term in the integrand in (7):

$$\int_0^{\infty} e^{-\rho t}\ln A(t)\,dt = \frac{\ln A_0}{\rho} + \frac{1}{\rho}\int_0^{\infty} e^{-\rho t}\frac{\dot{A}(t)}{A(t)}\,dt. \tag{16}$$

This formula is valid for any admissible trajectory $A(\cdot)$ of problem (P). To show this, it suffices first to integrate by parts on a finite time interval $[0, T]$ and then pass to the limit as $T \to \infty$:

$$\int_0^{T} e^{-\rho t}\ln A(t)\,dt = \frac{\ln A_0 - e^{-\rho T}\ln A(T)}{\rho} + \frac{1}{\rho}\int_0^{T} e^{-\rho t}\frac{\dot{A}(t)}{A(t)}\,dt. \tag{17}$$

Due to (10) the integral on the left-hand side and the first term on the right-hand side tend to the corresponding terms in (16). Further, $\dot{A}(t) \geq 0$, $t > 0$; therefore, $e^{-\rho t}\dot{A}(t)/A(t)$ is integrable on $[0, +\infty)$ and the last term in (17) tends to the last term in (16).

Substituting $\dot{A}(t)$ from (4) into (16), we obtain

$$\begin{aligned}
\int_0^{\infty} e^{-\rho t}\ln A(t)\,dt &= \frac{\ln A_0}{\rho} + \frac{1}{\rho}\int_0^{\infty} e^{-\rho t}\frac{A(t)^{\theta}[L^A(t)]^{\eta}v(t)^{1-\eta}S(t)^{1-\eta}}{A(t)}\,dt \\
&= \frac{\ln A_0}{\rho} + \frac{1}{\rho}\int_0^{\infty} e^{-\rho t}[L^A(t)]^{\eta}v(t)^{1-\eta}x(t)\,dt.
\end{aligned}$$

Similarly,

$$\begin{aligned}\int_0^T e^{-\rho t}\ln R_1(t)\,dt &= \int_0^T e^{-\rho t}\big[\ln u(t)+\ln S(t)\big]\,dt\\ &= \int_0^T e^{-\rho t}\ln u(t)\,dt + \frac{\ln S_0 - e^{-\rho T}\ln S(T)}{\rho} + \frac{1}{\rho}\int_0^T e^{-\rho t}\frac{\dot S(t)}{S(t)}\,dt\\ &= \frac{\ln S_0 - e^{-\rho T}\ln S(T)}{\rho} + \int_0^T e^{-\rho t}\left[\ln u(t) - \frac{u(t)+v(t)}{\rho}\right]dt.\end{aligned}$$

Passing to the limit as $T\to\infty$, we see that

$$\int_0^\infty e^{-\rho t}\ln R_1(t)\,dt = \frac{\ln S_0}{\rho} + \int_0^\infty e^{-\rho t}\left[\ln u(t) - \frac{u(t)+v(t)}{\rho}\right]dt,$$

where both sides may be $-\infty$.

Thus, multiplying $J(A(\cdot),L^A(\cdot),R_1(\cdot))$ by ρ and neglecting constant terms, we arrive at the functional

$$\begin{aligned}J_1(x(\cdot),L^A(\cdot),u(\cdot),v(\cdot)) = \int_0^\infty e^{-\rho t}\Big\{&\varkappa[L^A(t)]^\eta v(t)^{1-\eta}x(t) + \alpha\rho\ln[L-L^A(t)]\\ &+(1-\alpha)\rho\ln u(t) - (1-\alpha)[u(t)+v(t)]\Big\}\,dt. \qquad (18)\end{aligned}$$

Now consider the following optimal control problem (P1) (see (14), (15) and (18)):

$$\dot x(t) = -(1-\eta)[u(t)+v(t)]x(t) - (1-\theta)[L^A(t)]^\eta v(t)^{1-\eta}x(t)^2, \qquad (19)$$

$$v(t)\in[0,\infty), \qquad L^A(t)\in[0,L), \qquad u(t)\in(0,\infty), \qquad (20)$$

$$x(0)=x_0,$$

$$\begin{aligned}J_1(x(\cdot),L^A(\cdot),u(\cdot),v(\cdot)) = \int_0^\infty e^{-\rho t}\Big\{&\varkappa[L^A(t)]^\eta v(t)^{1-\eta}x(t) + \alpha\rho\ln[L-L^A(t)]\\ &+(1-\alpha)\rho\ln u(t) - (1-\alpha)[u(t)+v(t)]\Big\}\,dt \to \max. \qquad (21)\end{aligned}$$

We say that a control $\tilde w(\cdot) = (L^A(\cdot),u(\cdot),v(\cdot))\colon [0,\infty)\to[0,L)\times(0,\infty)\times[0,\infty)$ (which is a triple of measurable functions) is admissible in problem (P1) if the functions $u(\cdot)$ and $v(\cdot)$ are locally bounded. The corresponding trajectory $x(\cdot)\colon[0,\tau)\to\mathbb{R}^1$, $\tau>0$, can obviously be extended to the whole infinite time interval $[0,\infty)$. So, without loss of generality, we assume that any admissible trajectory $x(\cdot)$ is defined on $[0,\infty)$. A pair $(x(\cdot),w(\cdot))$ where $w(\cdot)$ is an admissible control and $x(\cdot)$ is the corresponding trajectory is called an admissible pair or a process in problem (P1).

Note that, structurally, problem (P1) is simpler than problem (P) because problem (P1) does not contain integral constraints on the control variables. Problem (P1) is equivalent to problem (P) in the following sense:

Lemma 1. *For fixed A_0 and S_0, there is a one-to-one correspondence between processes $(A(\cdot), w(\cdot))$ in problem* (P) *and $(x(\cdot), \tilde{w}(\cdot))$ in problem* (P1)*. Moreover, the corresponding values of the objective functionals $J(A(\cdot), L^A(\cdot), R_1(\cdot))$ and $J_1(x(\cdot), L^A(\cdot), u(\cdot), v(\cdot))$ are related by a linear transformation of the form*

$$J_1(x(\cdot), L^A(\cdot), u(\cdot), v(\cdot)) = \rho J(A(\cdot), L^A(\cdot), R_1(\cdot)) + C, \tag{22}$$

where C depends only on ρ, A_0 and S_0.

Proof. As shown above, any process $(A(\cdot), w(\cdot)) = (A(\cdot), L^A(\cdot), R_1(\cdot), R_2(\cdot))$ in problem (P) generates a process $(x(\cdot), \tilde{w}(\cdot)) = (x(\cdot), L^A(\cdot), u(\cdot), v(\cdot))$ in problem (P1), and relation (22) is valid for these processes.

Now, we show that any control process $(x(\cdot), \tilde{w}(\cdot)) = (x(\cdot), L^A(\cdot), u(\cdot), v(\cdot))$ in problem (P1) corresponds to a control process $(A(\cdot), w(\cdot)) = (A(\cdot), L^A(\cdot), R_1(\cdot), R_2(\cdot))$ in problem (P). First, using the controls $u(\cdot)$ and $v(\cdot)$, we determine $S(\cdot)$ as a unique solution to the Cauchy problem

$$\dot{S}(t) = -[u(t) + v(t)]S(t), \qquad S(0) = S_0.$$

Since $u(\cdot) + v(\cdot)$ is positive and locally bounded, we obtain a positive monotonically decreasing function $S(\cdot)$ defined on $[0, \infty)$. Then we define $R_1(t) = u(t)S(t)$ and $R_2(t) = v(t)S(t)$, $t \geq 0$, which are locally bounded and satisfy the integral constraint in (5). Finally, we find $A(\cdot)$ as a unique solution to the Cauchy problem

$$\frac{d}{dt}\big[A(t)^{1-\theta}\big] = (1-\theta)[L^A(t)]^{\eta} v(t)^{1-\eta} S(t)^{1-\eta}, \qquad A(0) = A_0$$

if $\theta < 1$, or as a unique solution to the Cauchy problem

$$\frac{d}{dt}\big[\ln A(t)\big] = [L^A(t)]^{\eta} v(t)^{1-\eta} S(t)^{1-\eta}, \qquad A(0) = A_0$$

if $\theta = 1$. This is certainly possible because the right-hand side of each of these equations is positive and locally bounded.

We thus have a process $(A(\cdot), w(\cdot)) = (A(\cdot), L^A(\cdot), R_1(\cdot), R_2(\cdot))$ in problem (P). Passing from this process $(A(\cdot), w(\cdot))$ in problem (P) back to some process $(x_1(\cdot), \tilde{w}_1(\cdot))$ in problem (P1) along the scheme described at the beginning of this section, we see that $\tilde{w}_1(\cdot) = \tilde{w}(\cdot)$ and $x_1(\cdot)$ satisfies the same Cauchy problem (14), (15) as $x(\cdot)$. Therefore, by the uniqueness theorem for solutions of differential equations, $x_1(\cdot) = x(\cdot)$. This proves the required one-to-one correspondence between the admissible processes in problems (P) and (P1). Since (22) holds for the processes $(A(\cdot), w(\cdot))$ and $(x_1(\cdot), \tilde{w}_1(\cdot))$, and $(x_1(\cdot), \tilde{w}_1(\cdot)) = (x(\cdot), \tilde{w}(\cdot))$, we conclude that (22) is valid for $(A(\cdot), w(\cdot))$ and $(x(\cdot), \tilde{w}(\cdot))$. □

As a direct consequence of Lemma 1 and estimate (8) we arrive at

Lemma 2. *There exists a constant $M_1 > 0$ depending only on ρ, L, A_0 and S_0 such that*

$$\sup_{(x(\cdot), \tilde{w}(\cdot))} \int_0^{\infty} e^{-\rho t} \Big\{ \varkappa [L^A(t)]^{\eta} v(t)^{1-\eta} x(t) + \alpha\rho \ln[L - L^A(t)] + (1-\alpha)\rho \ln u(t) - (1-\alpha)[u(t) + v(t)] \Big\}\, dt \leq M_1,$$

where the supremum is taken over all admissible pairs $(x(\cdot), \tilde{w}(\cdot))$ in problem (P1)*.*

Lemma 2 allows us to define an optimal control $\tilde{w}_*(\cdot)\colon [0,\infty) \to \mathbb{R}^3$ in problem (P1) as a welfare-maximizing triple $\tilde{w}_*(\cdot) = (L_*^A(\cdot), u_*(\cdot), v_*(\cdot))$. The corresponding admissible trajectory $x_*(\cdot)$ is an optimal one in problem (P1).

To recapitulate, we showed that a process $(A(\cdot), w(\cdot))$ is optimal in problem (P) if and only if the corresponding process $(x(\cdot), \tilde{w}(\cdot))$ is optimal in problem (P1). In the next section we formulate and prove two main theoretical results on which the subsequent solution of the problem is based.

4 Existence of an optimal control and Pontryagin's maximum principle

Denote

$$
\begin{aligned}
f(x,\ell,u,v) &= -(1-\eta)(u+v)x - (1-\theta)\ell^\eta v^{1-\eta} x^2,\\
g(x,\ell,u,v) &= \varkappa \ell^\eta v^{1-\eta} x + \alpha\rho \ln(L-\ell) + (1-\alpha)\rho \ln u - (1-\alpha)(u+v),\\
&\qquad x>0, \qquad \ell\in[0,L), \qquad u>0, \qquad v\geq 0,
\end{aligned}
\tag{23}
$$

so that (19) and (21) become

$$
\begin{gathered}
\dot{x}(t) = f(x(t), L^A(t), u(t), v(t)),\\
J_1(x(\cdot), L^A(\cdot), u(\cdot), v(\cdot)) = \int_0^\infty e^{-\rho t} g(x(t), L^A(t), u(t), v(t))\, dt \to \max.
\end{gathered}
$$

Let $\mathcal{M}(x,u,v,p)$ and $M(x,p)$ be the current value Hamilton–Pontryagin function and the current value Hamiltonian for problem (P1) in the normal form:

$$
\begin{aligned}
\mathcal{M}(x,\ell,u,v,p) &= f(x,\ell,u,v)p + g(x,\ell,u,v)\\
&= -(1-\eta)(u+v)xp - (1-\theta)\ell^\eta v^{1-\eta}x^2 p + \varkappa \ell^\eta v^{1-\eta} x\\
&\quad + \alpha\rho\ln(L-\ell) + (1-\alpha)\rho\ln u - (1-\alpha)(u+v),
\end{aligned}
\tag{24}
$$

$$
M(x,p) = \sup_{\ell\in[0,L),\, u>0,\, v\geq 0} \mathcal{M}(x,\ell,u,v,p).
$$

Here $x>0$, $\ell\in[0,L)$, $u>0$, $v\geq 0$ and $p\in\mathbb{R}^1$.

Next, we formulate two important theorems (an existence theorem and a version of the Pontryagin maximum principle for problem (P1)) that allow us to perform a qualitative analysis of the solution to problem (P) (in Section 5). The proofs of these theorems (together with all necessary auxiliary statements) constitute the rest of this section.

Theorem 1 (existence)**.** *There exists an optimal process* $(x_*(\cdot), \tilde{w}_*(\cdot))$ *in problem* (P1). *The process* $(A_*(\cdot), w_*(\cdot))$ *corresponding to* $(x_*(\cdot), \tilde{w}_*(\cdot))$ (*in the sense of Lemma* 1) *is optimal in problem* (P).

Theorem 2 (maximum principle). *Let $(x_*(\cdot), \tilde{w}_*(\cdot)) = (x_*(\cdot), L^A_*(\cdot), u_*(\cdot), v_*(\cdot))$ be an optimal process in problem* (P1) *and $(A_*(\cdot), w_*(\cdot))$ be the corresponding (in the sense of Lemma* 1*) optimal process in problem* (P)*. Then there exists a current value adjoint variable $p(\cdot)$ such that the following conditions hold*:

(i) *The process $(x_*(\cdot), \tilde{w}_*(\cdot))$, together with the current value adjoint variable $p(\cdot)$, satisfies the core relations of the Pontryagin maximum principle in the normal form on the infinite time interval* $[0, \infty)$:

$$\dot{p}(t) = \rho p(t) - \frac{\partial \mathcal{M}(x_*(t), L^A_*(t), u_*(t), v_*(t), p(t))}{\partial x} \qquad \text{for a.e. } t > 0, \tag{25}$$

$$\mathcal{M}(x_*(t), L^A_*(t), u_*(t), v_*(t), p(t)) = M(x_*(t), p(t)) \qquad \text{for a.e. } t > 0. \tag{26}$$

(ii) *The process $(x_*(\cdot), \tilde{w}_*(\cdot))$, together with the current value adjoint variable $p(\cdot)$, satisfies the normal-form stationarity condition*

$$M(x_*(t), p(t)) = \rho e^{\rho t} \int_t^\infty e^{-\rho s} g(x_*(s), L^A_*(s), u_*(s), v_*(s))\, ds \qquad \text{for all} \quad t \geq 0.$$

(iii) *For any $t \geq 0$*

$$p(t) = e^{\rho t} e^{-y(t)} \int_t^\infty e^{-\rho s} e^{y(s)} \frac{\partial g(x_*(s), L^A_*(s), u_*(s), v_*(s))}{\partial x}\, ds, \tag{27}$$

where $y(t) = \int_0^t \frac{\partial f(x_*(s), L^A_*(s), u_*(s), v_*(s))}{\partial x}\, ds \leq 0$.

Let us outline the scheme of proofs of these two theorems. First, we show that it suffices to consider only bounded controls in problem (P1). Then we introduce the problem with a slightly modified objective functional, which is defined for controls that take values in the compact closure of the admissible control set. We show that the optimal processes in these two problems coincide. Finally, using standard results of optimal control theory, we prove analogs of Theorems 1 and 2 for the modified problem, which automatically implies the assertions of Theorems 1 and 2. The above approach is presented as a series of auxiliary lemmas that are subsequently used to prove the theorems.

Denote

$$V_0 = \left(\frac{(1-\eta)\varkappa L^\eta x_0}{1-\alpha} \right)^{1/\eta} \tag{28}$$

and consider the following optimal control problem (P1$'$) with bounded controls:

$$\dot{x}(t) = -(1-\eta)[u(t) + v(t)]x(t) - (1-\theta)[L^A(t)]^\eta v(t)^{1-\eta} x(t)^2, \tag{29}$$

$$L^A(t) \in [0, L), \qquad u(t) \in (0, \rho], \qquad v(t) \in [0, V_0], \tag{30}$$

$$x(0) = x_0, \tag{31}$$

$$J_1(x(\cdot), L^A(\cdot), u(\cdot), v(\cdot)) = \int_0^\infty e^{-\rho t} \Big\{ \varkappa [L^A(t)]^\eta v(t)^{1-\eta} x(t) + \alpha\rho \ln[L - L^A(t)] + (1-\alpha)\rho \ln u(t) - (1-\alpha)[u(t) + v(t)] \Big\}\, dt \to \max. \tag{32}$$

Lemma 3. *If* $\tilde{w}_*(\cdot) = (L^A_*(\cdot), u_*(\cdot), v_*(\cdot))$ *is an optimal admissible control in problem* (P1), *then*

$$u_*(t) \le \rho \qquad \text{and} \qquad v_*(t) \le V_0 = \left(\frac{(1-\eta)\varkappa L^\eta x_0}{1-\alpha}\right)^{1/\eta} \qquad \text{for a.e. } t > 0,$$

and so $\tilde{w}_*(\cdot)$ *is also an optimal admissible control in problem* (P1′). *Conversely, if* $\hat{\tilde{w}}_*(\cdot)$ *is an optimal admissible control in problem* (P1′), *then it is also an optimal admissible control in problem* (P1).

Before proving the lemma, we point out a corollary to this lemma and formula (27).

Corollary 1. *The current value adjoint variable* $p(\cdot)$ *satisfying the conditions of Theorem 2 is bounded:*

$$0 \le p(t) \le \frac{\varkappa L^\eta V_0^{1-\eta}}{\rho} \qquad \text{for all } t > 0$$

(*if* $\eta = 1$, *then* $V_0 = 0$ *and we consider* $V_0^{1-\eta}$ *to be* 1). *In particular, the transversality condition*

$$\lim_{t\to\infty} e^{-\rho t} x_*(t) p(t) = 0$$

holds for any optimal process $(x_*(\cdot), \tilde{w}_*(\cdot))$ *in problem* (P1).

Proof. Indeed, since $\frac{\partial f}{\partial x}(x, \ell, u, v) \le 0$ for all $x > 0$, $\ell \in [0, L)$, $u > 0$ and $v \ge 0$, it follows that $y(\cdot)$ is a monotonically decreasing function, and so

$$0 \le p(t) \le e^{\rho t} \int_t^\infty e^{-\rho s} \varkappa L^A_*(s)^\eta v_*(s)^{1-\eta}\, ds \le \frac{\varkappa L^\eta V_0^{1-\eta}}{\rho} \qquad \text{for all } t > 0.$$

This implies the transversality condition, as $0 < x_*(t) \le x_0$ for $t > 0$. □

Proof of Lemma 3. Let $\tilde{w}(\cdot) = (L^A(\cdot), u(\cdot), v(\cdot))$ be an admissible control in problem (P1) such that $\operatorname{ess\,sup}_{t>0} u(t) > \rho$ or $\operatorname{ess\,sup}_{t>0} v(t) > V_0$. Define a new admissible bounded control $\overline{w}(\cdot) = (L^A(\cdot), \bar{u}(\cdot), \bar{v}(\cdot))$ with $\bar{u}(t) = \min\{u(t), \rho\}$ and $\bar{v}(t) = \min\{v(t), V_0\}$, $t \ge 0$. Note that $\overline{w}(\cdot)$ is also an admissible control in problem (P1′).

Let $x(\cdot)$ and $\bar{x}(\cdot)$ be the trajectories of problem (P1) (with the same initial condition x_0) that correspond to $\tilde{w}(\cdot)$ and $\overline{w}(\cdot)$, respectively ($\bar{x}(\cdot)$ is also a trajectory of problem (P1′)). Then we have

$$\bar{u}(t) \le u(t), \qquad \bar{v}(t) \le v(t) \qquad \text{and} \qquad x_0 \ge \bar{x}(t) \ge x(t) > 0 \qquad \text{for all } t > 0$$

by virtue of equation (19). Therefore,

$$\begin{aligned} J_1(x(\cdot), L^A(\cdot), u(\cdot), v(\cdot)) &\le \int_0^\infty e^{-\rho t}\Big\{\varkappa[L^A(t)]^\eta v(t)^{1-\eta}\bar{x}(t) + \alpha\rho\ln[L - L^A(t)] \\ &\qquad + (1-\alpha)\rho\ln u(t) - (1-\alpha)[u(t) + v(t)]\Big\}\, dt \\ &< \int_0^\infty e^{-\rho t}\Big\{\varkappa[L^A(t)]^\eta \bar{v}(t)^{1-\eta}\bar{x}(t) + \alpha\rho\ln[L - L^A(t)] \\ &\qquad + (1-\alpha)\rho\ln \bar{u}(t) - (1-\alpha)[\bar{u}(t) + \bar{v}(t)]\Big\}\, dt \\ &= J_1(\bar{x}(\cdot), L^A(\cdot), \bar{u}(\cdot), \bar{v}(\cdot)), \end{aligned}$$

where we used the inequalities

$$\frac{d}{du}\big((1-\alpha)\rho\ln u-(1-\alpha)u\big)<0,\qquad \frac{d}{dv}\big(\varkappa[L^A(t)]^\eta v^{1-\eta}\bar{x}(t)-(1-\alpha)v\big)<0$$

for all $t>0$ and $u>\rho$, $v>V_0$.

Thus, we see that if $\operatorname{ess\,sup}_{t>0}u(t)>\rho$ or $\operatorname{ess\,sup}_{t>0}v(t)>V_0$, then the control $\tilde{w}(\cdot)$ cannot be optimal. This proves the first part of the lemma.

Conversely, if $(x_*(\cdot),\tilde{w}_*(\cdot))=(x_*(\cdot),L^A_*(\cdot),u_*(\cdot),v_*(\cdot))$ is an optimal process in problem (P1′) and $(x(\cdot),\tilde{w}(\cdot))=(x(\cdot),L^A(\cdot),u(\cdot),v(\cdot))$ is any process in problem (P1), then, again, introducing a new bounded control $\overline{w}(\cdot)=(L^A(\cdot),\bar{u}(\cdot),\bar{v}(\cdot))$ with $\bar{u}(t)=\min\{u(t),\rho\}$ and $\bar{v}(t)=\min\{v(t),V_0\}$, $t\geq 0$, we see that

$$J_1(x(\cdot),L^A(\cdot),u(\cdot),v(\cdot))\leq J_1(\bar{x}(\cdot),L^A(\cdot),\bar{u}(\cdot),\bar{v}(\cdot))\leq J_1(x_*(\cdot),L^A_*(\cdot),u_*(\cdot),v_*(\cdot)),$$

where $\bar{x}(\cdot)$ is the trajectory of problem (P1) (as well as of (P1′)) corresponding to the control $\overline{w}(\cdot)$. □

Our next goal is to establish the existence of an optimal admissible control $\tilde{w}_*(\cdot)$ in problem (P1′). To apply a standard existence theorem of optimal control theory, we need to compactify the range of values of the control variables. For this purpose, we introduce the function

$$\mathcal{L}_\varepsilon(\xi)=\begin{cases}\ln\varepsilon+\dfrac{1}{\varepsilon}(\xi-\varepsilon) & \text{for }\ 0\leq\xi\leq\varepsilon,\\ \ln\xi & \text{for }\ \xi>\varepsilon,\end{cases}\tag{33}$$

where $\varepsilon<1$ is a small positive constant, to the utility functional $J_1(x(\cdot),L^A(\cdot),u(\cdot),v(\cdot))$. Obviously, $\mathcal{L}_\varepsilon(\cdot)$ is a continuously differentiable concave function on $[0,\infty)$ and $\mathcal{L}_\varepsilon(\xi)\geq\ln\xi$ for $\xi\in(0,\infty)$.

Now consider an auxiliary problem (P_ε):

$$\dot{x}(t)=-(1-\eta)[u(t)+v(t)]x(t)-(1-\theta)[L^A(t)]^\eta v(t)^{1-\eta}x(t)^2,\tag{34}$$

$$L^A(t)\in[0,L],\qquad u(t)\in[0,\rho],\qquad v(t)\in[0,V_0],\tag{35}$$

$$x(0)=x_0,$$

$$\begin{aligned}J_\varepsilon(x(\cdot),L^A(\cdot),u(\cdot),v(\cdot))=\int_0^\infty e^{-\rho t}\Big\{&\varkappa[L^A(t)]^\eta v(t)^{1-\eta}x(t)+\alpha\rho\mathcal{L}_\varepsilon\big(L-L^A(t)\big)\\&+(1-\alpha)\rho\mathcal{L}_\varepsilon(u(t))-(1-\alpha)[u(t)+v(t)]\Big\}\,dt\to\max,\end{aligned}\tag{36}$$

where x_0 is the same as in (31). Clearly, any process $(x(\cdot),\tilde{w}(\cdot))=(x(\cdot),L^A(\cdot),u(\cdot),v(\cdot))$ in problem (P1′) is also an admissible process in problem (P_ε).

Lemma 4. *If there is an optimal process* $(x_*(\cdot),\tilde{w}_*(\cdot))=(x_*(\cdot),L^A_*(\cdot),u_*(\cdot),v_*(\cdot))$ *in problem* (P_ε) *such that* $L^A_*(t)\leq L-\varepsilon$ *and* $u_*(t)\geq\varepsilon$ *for a.e.* $t\in(0,\infty)$, *then*

(i) *this process is also optimal in problem* (P1′);

(ii) *any other optimal process* $(\hat{x}_*(\cdot),\hat{\tilde{w}}_*(\cdot))=(\hat{x}_*(\cdot),\hat{L}^A_*(\cdot),\hat{u}_*(\cdot),\hat{v}_*(\cdot))$ *(if it exists) in problem* (P1′) *is such that* $\hat{L}^A_*(t)\leq L-\varepsilon$ *and* $\hat{u}_*(t)\geq\varepsilon$ *for a.e.* $t\in(0,\infty)$ *and so it is also optimal in problem* (P_ε).

Proof. Assertion (i) is valid because $J_\varepsilon(x(\cdot), L^A(\cdot), u(\cdot), v(\cdot)) \geq J_1(x(\cdot), L^A(\cdot), u(\cdot), v(\cdot))$ for any admissible process $(x(\cdot), \tilde{w}(\cdot)) = (x(\cdot), L^A(\cdot), u(\cdot), v(\cdot))$ in problem (P1′), while $J_\varepsilon(x_*(\cdot), L_*^A(\cdot), u_*(\cdot), v_*(\cdot)) = J_1(x_*(\cdot), L_*^A(\cdot), u_*(\cdot), v_*(\cdot))$.

If $(\hat{x}(\cdot), \hat{\tilde{w}}(\cdot)) = (\hat{x}(\cdot), \hat{L}^A(\cdot), \hat{u}(\cdot), \hat{v}(\cdot))$ is a process in problem (P1′) such that $\hat{L}^A(t) > L - \varepsilon$ or $\hat{u}(t) < \varepsilon$ on a positive measure set of values of t, then

$$\begin{aligned} J_1(\hat{x}(\cdot), \hat{L}^A(\cdot), \hat{u}(\cdot), \hat{v}(\cdot)) &< J_\varepsilon(\hat{x}(\cdot), \hat{L}^A(\cdot), \hat{u}(\cdot), \hat{v}(\cdot)) \leq J_\varepsilon(x_*(\cdot), L_*^A(\cdot), u_*(\cdot), v_*(\cdot)) \\ &= J_1(x_*(\cdot), L_*^A(\cdot), u_*(\cdot), v_*(\cdot)) \end{aligned}$$

and hence this process cannot be optimal in problem (P1′). This implies (ii). $\square$

Denote

$$W = [0, L] \times [0, \rho] \times [0, V_0]$$

and

$$\begin{gathered} g_\varepsilon(x, \ell, u, v) = \varkappa \ell^\eta v^{1-\eta} x + \alpha\rho\mathcal{L}_\varepsilon(L - \ell) + (1-\alpha)\rho\mathcal{L}_\varepsilon(u) - (1-\alpha)(u + v), \\ x > 0, \qquad (\ell, u, v) \in W, \end{gathered} \tag{37}$$

so that (34) and (36) become

$$\begin{aligned} \dot{x}(t) &= f(x(t), L^A(t), u(t), v(t)), \\ J_\varepsilon(x(\cdot), L^A(\cdot), u(\cdot), v(\cdot)) &= \int_0^\infty e^{-\rho t} g_\varepsilon(x(t), L^A(t), u(t), v(t))\, dt \to \max \end{aligned}$$

(see (23)).

For every $x > 0$, consider the following set, which is standard in optimal control theory:

$$Q(x) = \big\{(z^0, z) \in \mathbb{R}^2 : \ z^0 \leq g_\varepsilon(x, \ell, u, v), \ z = f(x, \ell, u, v), \ (\ell, u, v) \in W\big\}.$$

Lemma 5. *For every $x > 0$, the set $Q(x)$ is convex.*

Proof. It suffices to show that for any two points $(z_1^0, z_1), (z_2^0, z_2) \in Q(x)$ the midpoint of the segment joining (z_1^0, z_1) to (z_2^0, z_2) also lies in $Q(x)$. Let $z_i = f(x, \ell_i, u_i, v_i)$ and $z_i^0 \leq g_\varepsilon(x, \ell_i, u_i, v_i)$ for some $(\ell_i, u_i, v_i) \in W$ $(i = 1, 2)$. We need to show that there exists $(\bar{\ell}, \bar{u}, \bar{v}) \in W$ such that

$$f(x, \bar{\ell}, \bar{u}, \bar{v}) = \bar{z} = \frac{z_1 + z_2}{2} \qquad \text{and} \qquad g_\varepsilon(x, \bar{\ell}, \bar{u}, \bar{v}) \geq \bar{z}^0 = \frac{z_1^0 + z_2^0}{2}.$$

We will seek $(\bar{\ell}, \bar{u}, \bar{v})$ in the form

$$\bar{\ell} = \bar{\ell}(\epsilon) = \frac{\ell_1 + \ell_2}{2} - \epsilon, \qquad \bar{u} = \frac{u_1 + u_2}{2}, \qquad \bar{v} = \frac{v_1 + v_2}{2}$$

with $0 \leq \epsilon \leq \frac{\ell_1 + \ell_2}{2}$. It is obvious that such a triple belongs to W.

Note that

$$\left(\frac{\ell_1+\ell_2}{2}\right)^{\eta}\left(\frac{v_1+v_2}{2}\right)^{1-\eta} \geq \frac{\ell_1^{\eta}v_1^{1-\eta}+\ell_2^{\eta}v_2^{1-\eta}}{2}, \qquad 0\leq\eta\leq 1$$

(see, e.g., [11, Theorem 38]). Therefore,

$$f(x,0,\bar{u},\bar{v}) \geq \bar{z} \qquad \text{and} \qquad f(x,\bar{\ell}(0),\bar{u},\bar{v}) \leq \bar{z}.$$

Since $f(x,\bar{\ell}(\cdot),\bar{u},\bar{v})$ is a continuous function of ϵ, there indeed exists an ϵ, $0\leq\epsilon\leq\frac{\ell_1+\ell_2}{2}$, such that

$$f(x,\bar{\ell}(\epsilon),\bar{u},\bar{v}) = \bar{z}. \tag{38}$$

We fix such an ϵ and write simply $\bar{\ell}$ instead of $\bar{\ell}(\epsilon)$ in what follows.

Now let us show that $g_\varepsilon(x,\bar{\ell},\bar{u},\bar{v}) \geq \bar{z}^0$. Note that due to (38), for $\theta<1$,

$$\begin{aligned}\bar{\ell}^{\eta}\bar{v}^{1-\eta}x &= \frac{-(1-\eta)(\bar{u}+\bar{v})x-\bar{z}}{(1-\theta)x} = \frac{-(1-\eta)(u_1+u_2+v_1+v_2)x-(z_1+z_2)}{2(1-\theta)x}\\ &= \frac{\ell_1^{\eta}v_1^{1-\eta}x+\ell_2^{\eta}v_2^{1-\eta}x}{2}.\end{aligned} \tag{39}$$

If $\theta=1$, then $f(\cdot)$ does not depend on ℓ and so (38) holds for all ϵ. Therefore, choosing an appropriate ϵ, we can achieve the equality of the first and last expressions in the chain (39) in this case as well.

Since $\mathcal{L}_\varepsilon(\cdot)$ is a concave increasing function, we have $\mathcal{L}_\varepsilon(L-\bar{\ell}) \geq \mathcal{L}_\varepsilon(L-\bar{\ell}(0))$ and in view of (39) find that

$$g_\varepsilon(x,\bar{\ell},\bar{u},\bar{v}) \geq \frac{g_\varepsilon(x,\ell_1,u_1,v_1)+g_\varepsilon(x,\ell_2,u_2,v_2)}{2} \geq \bar{z}^0.$$

This completes the proof of Lemma 5. □

Lemma 6. *For any* ε, $0<\varepsilon<1$, *there exists an optimal control in problem* (P_ε). *Moreover, if* ε *is small enough, then any optimal control* $\tilde{w}(\cdot)=(L_*^A(\cdot),u_*(\cdot),v_*(\cdot))$ *in problem* (P_ε) *is such that* $L_*^A(t)\leq L-\varepsilon$ *and* $u_*(t)\geq\varepsilon$ *for a.e.* $t\in(0,\infty)$.

Proof. The existence follows from Theorem 2.1 in [4] and Lemma 5.

Note that problem (P_ε) falls within the case of dominating discount (see [4, Section 12]), so we can apply the version of Pontryagin's maximum principle formulated in [4, Theorem 12.1] to this problem. To this end, define the current value Hamilton–Pontryagin function $\mathcal{M}_\varepsilon(x,u,v,p)$ and the current value Hamiltonian $M_\varepsilon(x,p)$ in problem (P_ε) in the normal form:

$$\begin{aligned}\mathcal{M}_\varepsilon(x,\ell,u,v,p) &= f(x,\ell,u,v)p+g_\varepsilon(x,\ell,u,v)\\ &= -(1-\eta)(u+v)xp-(1-\theta)\ell^{\eta}v^{1-\eta}x^2p+\varkappa\ell^{\eta}v^{1-\eta}x\\ &\quad +\alpha\rho\mathcal{L}_\varepsilon(L-\ell)+(1-\alpha)\rho\mathcal{L}_\varepsilon(u)-(1-\alpha)(u+v),\end{aligned} \tag{40}$$

$$M_\varepsilon(x,p) = \sup_{(\ell,u,v)\in W}\mathcal{M}_\varepsilon(x,\ell,u,v,p). \tag{41}$$

Here $x>0$, $(\ell,u,v)\in W$ and $p\in\mathbb{R}^1$.

Let $(x_*(\cdot), \tilde{w}_*(\cdot)) = (x_*(\cdot), L_*^A(\cdot), u_*(\cdot), v_*(\cdot))$ be an optimal process in problem (P_ε). Then, by Theorem 12.1 from [4], we have

$$\mathcal{M}_\varepsilon(x_*(t), L_*^A(t), u_*(t), v_*(t), p(t)) = M_\varepsilon(x_*(t), p(t)) \qquad \text{for a.e. } t > 0, \tag{42}$$

where

$$p(t) = e^{\rho t} e^{-y(t)} \int_t^\infty e^{-\rho s} e^{y(s)} \frac{\partial g_\varepsilon(x_*(s), L_*^A(s), u_*(s), v_*(s))}{\partial x} \, ds \tag{43}$$

with the same $y(\cdot)$ as in Theorem 2. As shown in the proof of Corollary 1, $y(\cdot)$ is a monotonically decreasing function, and so

$$0 \le p(t) \le \frac{1}{\rho} \sup_{x>0,\ (\ell,u,v)\in W} \frac{\partial g_\varepsilon(x, \ell, u, v)}{\partial x} = \frac{\varkappa L^\eta V_0^{1-\eta}}{\rho} \qquad \text{for all } t > 0.$$

We also have $0 < x_*(\cdot) \le x_0$. However, it is easy to show that if ε is sufficiently small,[1] then the maximum of the function $\mathcal{M}_\varepsilon(x, \cdot, \cdot, \cdot, p)$ with respect to $(\ell, u, v) \in W$ for fixed $x \in (0, x_0]$ and $p \in [0, \varkappa L^\eta V_0^{1-\eta}/\rho]$ cannot be attained at a point (ℓ, u, v) such that $\ell > L - \varepsilon$ or $u < \varepsilon$. Indeed, it suffices to calculate the partial derivatives of $\mathcal{M}_\varepsilon$ with respect to ℓ and u.

This fact, together with the maximum condition (42), completes the proof of the lemma. $\square$

Proof of Theorem 1. Above we have shown that the auxiliary problem (P_ε) has a solution, i.e. an optimal process $(x_*(\cdot), \tilde{w}_*(\cdot)) = (x_*(\cdot), L_*^A(\cdot), u_*(\cdot), v_*(\cdot))$, and proved certain estimates for the corresponding optimal control (Lemma 6). These estimates show (Lemma 4) that any such solution is also an optimal process in problem (P1′), and so is an optimal process in problem (P1) (Lemma 3), which is equivalent to the original problem (P) (Lemma 1). Thus, we obtain the existence of an optimal control in problem (P). $\square$

Proof of Theorem 2. Fix a sufficiently small ε. By Lemmas 6 and 4(ii), $L_*^A(t) \le L - \varepsilon$ and $u_*(t) \ge \varepsilon$ for a.e. $t \in (0, \infty)$, and $(x_*(\cdot), \tilde{w}_*(\cdot))$ is an optimal process in problem (P_ε).

By Theorem 12.1 in [4], such an adjoint variable $p(\cdot)$ satisfying properties (i)–(iii) of Theorem 2 (with $g_\varepsilon(\cdot)$, $M_\varepsilon(\cdot)$ and $\mathcal{M}_\varepsilon(\cdot)$ instead of $g(\cdot)$, $M(\cdot)$ and $\mathcal{M}(\cdot)$, respectively) exists for the optimal process $(x_*(\cdot), \tilde{w}_*(\cdot))$ in problem (P_ε). Since $L_*^A(t) \le L - \varepsilon$ and $u_*(t) \ge \varepsilon$ for a.e. $t > 0$, we have $g(x_*(t), L_*^A(t), u_*(t), v_*(t)) = g_\varepsilon(x_*(t), L_*^A(t), u_*(t), v_*(t))$ and $\mathcal{M}(x_*(t), L_*^A(t), u_*(t), v_*(t), p(t)) = \mathcal{M}_\varepsilon(x_*(t), L_*^A(t), u_*(t), v_*(t), p(t))$ for a.e. $t > 0$. Moreover, since $\mathcal{M}(x, \ell, u, v, p) \le \mathcal{M}_\varepsilon(x, \ell, u, v, p)$ for all $x > 0$, $p > 0$ and $(\ell, u, v) \in W$, we also have $M(x_*(t), p(t)) = M_\varepsilon(x_*(t), p(t))$.

Thus, properties (i)–(iii) of Theorem 2 with $g(\cdot)$, $M(\cdot)$ and $\mathcal{M}(\cdot)$ follow from the same properties with $g_\varepsilon(\cdot)$, $M_\varepsilon(\cdot)$ and $\mathcal{M}_\varepsilon(\cdot)$. In particular, (42) and (43) become (26) and (27). $\square$

Theorem 2 allows us to explicitly write the Hamiltonian system of the Pontryagin maximum principle for problem (P1). In the next section, we will analyze the qualitative behavior of solutions to this system and single out all optimal regimes.

[1] Of course, the upper bound for ε that guarantees the validity of this statement depends on x_0, but x_0 is fixed from the onset.

5 Analysis of the Hamiltonian system

We know from Theorem 1 that an optimal process $(x_*(\cdot), \tilde{w}_*(\cdot))$ in problem (P1) exists and satisfies the relations of Theorem 2. Using Theorem 2, we can construct the Hamiltonian system of the Pontryagin maximum principle for problem (P1) in the variables $x(\cdot)$ and $p(\cdot)$ directly. However, to simplify the further analysis, we pass from the variable $p(\cdot)$ to a new variable $\phi(\cdot)$ defined as $\phi(t) = x(t)p(t)$, $t > 0$. Then we write and analyze the relations of the Hamiltonian system of the Pontryagin maximum principle for problem (P1) in the variables $x(\cdot)$ and $\phi(\cdot)$.

In terms of the variable $\phi(\cdot)$, the adjoint system (see (25)) and the maximum condition (see (26)) take the forms

$$\dot{\phi}(t) = \dot{x}(t)p(t) + x(t)\dot{p}(t) = \rho\phi(t) + L^A(t)^\eta v(t)^{1-\eta} x(t)\big[(1-\theta)\phi(t) - \varkappa\big] \tag{44}$$

and

$$\tilde{\mathcal{M}}(x, \ell, u, v, \phi) \to \max_{\ell\in[0,L), u>0, v\geq 0}, \tag{45}$$

respectively. Here the function $\tilde{\mathcal{M}}(\cdot)$ is defined by the equality (see (24))

$$\begin{aligned}\tilde{\mathcal{M}}(x, \ell, u, v, \phi) = &-\big[1-\alpha+(1-\eta)\phi\big](u+v)\\ &+\big[\varkappa-(1-\theta)\phi\big]\ell^\eta v^{1-\eta}x + \alpha\rho\ln(L-\ell) + (1-\alpha)\rho\ln u,\end{aligned} \tag{46}$$

for all $x > 0$, $\phi \geq 0$, $u > 0$, $v \geq 0$ and $0 \leq \ell < L$.

Our first aim is to write the Hamiltonian system of the maximum principle for problem (P1) in terms of the variables $x(\cdot)$ and $\phi(\cdot)$ by combining equations (19) and (44) (and using maximum condition (45)). To this end, we first express the quantities $L^A(x,\phi)$, $u(x,\phi)$ and $v(x,\phi)$ as functions of x and ϕ that are (unique) maximizers of $\tilde{\mathcal{M}}(\cdot)$ with respect to ℓ, u and v, respectively (see maximum condition (45)), for all $x > 0$ and $\phi \geq 0$. Then, substituting these maximizers into equations (19) and (44), we get the Hamiltonian system of the maximum principle for problem (P1) in the form

$$\begin{aligned}\dot{x}(t) = &-(1-\eta)[u(x(t),\phi(t)) + v(x(t),\phi(t))]x(t)\\ &-(1-\theta)L^A(x(t),\phi(t))^\eta v(x(t),\phi(t))^{1-\eta}x(t)^2,\\ \dot{\phi}(t) = &\ \rho\phi(t) + L^A(x(t),\phi(t))^\eta v(x(t),\phi(t))^{1-\eta}x(t)\big[(1-\theta)\phi(t)-\varkappa\big].\end{aligned} \tag{47}$$

The value $u(x,\phi)$ at which the maximum of $\tilde{\mathcal{M}}(\cdot)$ with respect to u is attained can easily be found by means of differentiation (see (46)):

$$u(x,\phi) = \frac{(1-\alpha)\rho}{1-\alpha+(1-\eta)\phi}. \tag{48}$$

If $\varkappa \leq (1-\theta)\phi$, then the maximum of $\tilde{\mathcal{M}}(\cdot)$ with respect to ℓ and v is attained for $v(x,\phi) = L^A(x,\phi) = 0$.

Suppose that $\varkappa > (1-\theta)\phi$. If $\eta = 1$, then $v(x, \phi) = 0$ simply because of the constraint $0 \le v \le V_0 = 0$ (see (35) and (28)), and $u(x, \phi) = \rho$ (see (48)). In this case it is obvious that the maximum point of $\tilde{\mathcal{M}}(\cdot)$ as a function of ℓ is given by

$$L^A(x, \phi) = L - \frac{\alpha\rho}{(\varkappa - (1-\theta)\phi)x}. \tag{49}$$

Finally, consider the case when $\varkappa > (1-\theta)\phi$ and $\eta < 1$. Note that $\tilde{\mathcal{M}}(x, \ell, u, v, \phi) \to -\infty$ as $v \to \infty$ or $\ell \to L - 0$. On the other hand, if one of the variables, v or ℓ, is zero, then the maximum with respect to the other variable is attained at zero. Therefore, the maximum of $\tilde{\mathcal{M}}(\cdot)$ with respect to ℓ and v is attained either at the point $v(x, \phi) = L^A(x, \phi) = 0$ or at an interior point, in which case this point can be found by equating the partial derivatives of $\tilde{\mathcal{M}}(\cdot)$ with respect to ℓ and v to zero:

$$\eta\big[\varkappa - (1-\theta)\phi\big]\Big(\frac{v}{\ell}\Big)^{1-\eta} x = \frac{\alpha\rho}{L-\ell}, \tag{50}$$

$$(1-\eta)\big[\varkappa - (1-\theta)\phi\big]\left(\frac{\ell}{v}\right)^{\eta} x = 1 - \alpha + (1-\eta)\phi. \tag{51}$$

Denoting

$$h(x, \phi) = \frac{1 - \alpha + (1-\eta)\phi}{(1-\eta)x[\varkappa - (1-\theta)\phi]}, \qquad x > 0, \quad 0 \le \phi < \frac{\varkappa}{1-\theta},$$

we find

$$\frac{\ell}{v} = h(x, \phi)^{\frac{1}{\eta}} \tag{52}$$

and

$$\ell = L - \frac{\alpha\rho h(x, \phi)^{\frac{1-\eta}{\eta}}}{\eta[\varkappa - (1-\theta)\phi]x} = L - \frac{\alpha\rho(1 - \alpha + (1-\eta)\phi)^{\frac{1-\eta}{\eta}}}{\eta(1-\eta)^{\frac{1-\eta}{\eta}}(x[\varkappa - (1-\theta)\phi])^{\frac{1}{\eta}}}, \tag{53}$$

$$v = \frac{L}{h(x, \phi)^{\frac{1}{\eta}}} - \frac{\alpha\rho h(x, \phi)^{-1}}{\eta[\varkappa - (1-\theta)\phi]x} = \frac{L((1-\eta)x[\varkappa - (1-\theta)\phi])^{\frac{1}{\eta}}}{(1 - \alpha + (1-\eta)\phi)^{\frac{1}{\eta}}} - \frac{\alpha\rho(1-\eta)}{\eta(1 - \alpha + (1-\eta)\phi)}. \tag{54}$$

If these formulas yield positive values $v(x, \phi)$ and $L^A(x, \phi)$ of v and ℓ, then this is the maximum point of $\tilde{\mathcal{M}}(\cdot)$ with respect to v and ℓ. Otherwise, the maximum point is $v(x, \phi) = L^A(x, \phi) = 0$.

Note that (53) and (54) for $\eta = 1$ turn into (49) and $v(x, \phi) = 0$, respectively, if we consider $(1-\eta)^{1-\eta}$ to be 1 for $\eta = 1$.

Set

$$h_1(\phi) = \frac{\alpha^\eta \rho^\eta (1 - \alpha + (1-\eta)\phi)^{1-\eta}}{L^\eta \eta^\eta (1-\eta)^{1-\eta}[\varkappa - (1-\theta)\phi]}, \qquad 0 \le (1-\theta)\phi < \varkappa,$$

and introduce the following sets (see Fig. 1):

$$\Gamma = \{(x, \phi) \in \mathbb{R}^2\colon\ x > 0,\ \phi \ge 0\},$$

$$\Gamma_0 = \big\{(x, \phi) \in \Gamma\colon\ (1-\theta)\phi \ge \varkappa \ \text{ or } \ \{(1-\theta)\phi < \varkappa,\ x < h_1(\phi)\}\big\}, \qquad \Gamma_1 = \Gamma \setminus \Gamma_0.$$

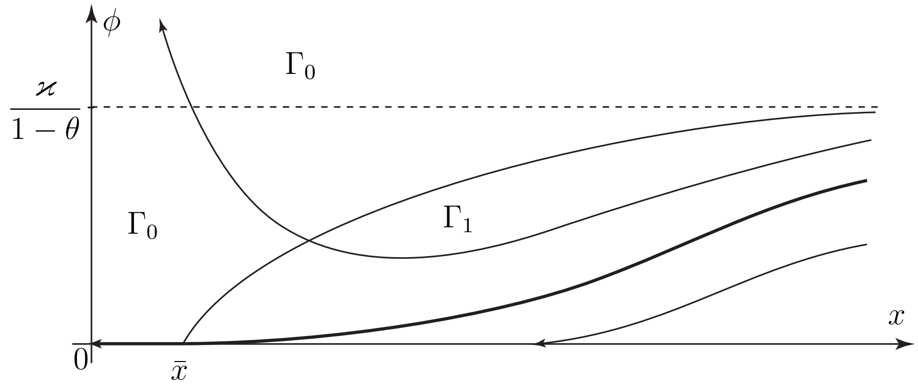

Figure 1: The sets Γ_0 and Γ_1 and the optimal trajectory (thick line). All trajectories lying above the optimal one tend to infinity along the ϕ-axis. All trajectories lying below the optimal one transversally intersect the x-axis.

According to the above analysis, in Γ_0 both $L^A(x,\phi)$ and $v(x,\phi)$ vanish, and so our Hamiltonian system (47) in Γ_0 has the form

$$\dot{x}(t) = -\frac{(1-\eta)(1-\alpha)\rho}{1-\alpha+(1-\eta)\phi(t)}x(t),$$
$$\dot{\phi}(t) = \rho\phi(t).$$

Note that $h_1(\cdot)$ is a monotonically increasing function of ϕ (except for the case $\eta=\theta=1$, in which $h_1(\cdot)\equiv \text{const}$). Therefore, any trajectory of our system that reaches the set Γ_0 cannot leave this set afterwards. (Indeed, at every point of Γ_0 we have $\dot{x}(\cdot)\le 0$ and $\dot{\phi}(\cdot)\ge 0$.) However, we know that $\phi(\cdot)$ is bounded along an optimal trajectory (e.g., by Corollary 1); hence the only candidate for an optimal trajectory in Γ_0 lies on the x-axis and looks like

$$x(t) = \bar{x}e^{-(1-\eta)\rho(t-\tau)}, \qquad \phi(t)=0 \qquad \text{for} \quad t\ge\tau, \tag{55}$$

where

$$\bar{x} = h_1(0) = \frac{\rho^\eta\alpha^\eta(1-\alpha)^{1-\eta}}{L^\eta\eta^\eta(1-\eta)^{1-\eta}\varkappa}. \tag{56}$$

On the other hand, since $\dot{x}(t)\le 0$, any bounded trajectory must tend to a fixed point. If $\eta<1$, then $\dot{x}(\cdot)<0$ in the interior of Γ_1 and consequently any trajectory of our system starting in Γ_1 eventually enters the set Γ_0. This shows that there is a unique bounded trajectory of our system, and hence the optimal process in problem (P1) is also unique. The tail of this trajectory is described by (55).

If $\eta=1$ and $\theta<1$, then for similar reasons any bounded trajectory starting in Γ_1 tends to the point $(\bar{x},0)$ on the boundary of Γ_1. Let us show that there is only one such trajectory $(\tilde{x}(\cdot),\tilde{\phi}(\cdot))$ in Γ_1. Indeed, if there were two trajectories lying in Γ_1 and tending to $(\bar{x},0)$, then any trajectory lying between these two would also tend to $(\bar{x},0)$ (because $\dot{x}(\cdot)\le 0$). However, this is impossible, as we can show, for example, by considering the linearization of the Hamiltonian system of the maximum principle in Γ_1 at the point $(\bar{x},0)$ and applying the Grobman–Hartman theorem (see [12]).

Finally, if $\eta = \theta = 1$, then $x(t) \equiv 1$ (see (12)) and $\dot{\phi}(t) = \rho\phi(t) - \ell\varkappa$, where $\ell = \max\{0, L - \frac{\alpha\rho}{\varkappa}\}$. Thus, the only bounded trajectory is the fixed point $x = 1$, $\phi = \max\{0, \frac{L\varkappa}{\rho} - \alpha\}$. Recall that in this case the optimal controls are $u(t) \equiv \rho$, $v(t) \equiv 0$ and $L^A(t) \equiv \max\{0, L - \frac{\alpha\rho}{\varkappa}\}$.

Let us now examine the initial part of the optimal trajectory lying in Γ_1, for $\eta < 1$. Using formulas (52) and (54), we find

$$\ell^\eta v^{1-\eta} = h(x,\phi)v = \frac{L}{h(x,\phi)^{\frac{1-\eta}{\eta}}} - \frac{\alpha\rho}{\eta x[\varkappa - (1-\theta)\phi]}.$$

Similarly, due to (48) and (54), we obtain

$$u + v = \frac{(\eta - \alpha)\rho}{\eta(1 - \alpha + (1-\eta)\phi)} + \frac{L}{h(x,\phi)^{\frac{1}{\eta}}}.$$

Thus, our system (47) in Γ_1 has the form

$$\begin{aligned} \dot{x}(t) &= -(1-\eta)\left[\frac{(\eta-\alpha)\rho}{\eta(1-\alpha+(1-\eta)\phi(t))} + \frac{L}{h(x(t),\phi(t))^{\frac{1}{\eta}}}\right]x(t) \\ &\quad - (1-\theta)\left[\frac{L}{h(x(t),\phi(t))^{\frac{1-\eta}{\eta}}} - \frac{\alpha\rho}{\eta x(t)[\varkappa - (1-\theta)\phi(t)]}\right]x(t)^2, \\ \dot{\phi}(t) &= \rho\phi(t) - \frac{L(1-\alpha+(1-\eta)\phi(t))}{(1-\eta)h(x(t),\phi(t))^{\frac{1}{\eta}}} + \frac{\alpha\rho}{\eta}, \end{aligned} \tag{57}$$

and we are interested in the trajectory $(\tilde{x}(\cdot), \tilde{\phi}(\cdot))$ that passes through the point $(\bar{x}, 0)$. It would be difficult to solve this system analytically, but for numerical simulations it suffices to know that the sought trajectory $(\tilde{x}(\cdot), \tilde{\phi}(\cdot))$ is a solution to the Cauchy problem for system (57) in reverse time (i.e., with the right-hand side taken with the opposite sign) under the initial condition $\tilde{x}(0) = \bar{x}$, $\tilde{\phi}(0) = 0$.

Moreover, since $\dot{\tilde{x}}(t) < 0$ for all $t > 0$, we can express $\tilde{\phi}(\cdot)$ as a function of $\tilde{x}(\cdot)$ along this trajectory, $\tilde{\phi} = \phi_*(x)$.

If $\eta = 1$ and $\theta < 1$, we can also express $\tilde{\phi}(\cdot)$ as a (continuous) function of $\tilde{x}(\cdot)$ along this trajectory, $\tilde{\phi} = \phi_*(x)$ (with $\phi_*(x) = 0$ for $x \le \bar{x}$). However, this trajectory cannot be found as a solution of the Cauchy problem, as described above, as $(\bar{x}, 0)$ is a fixed point of the Hamiltonian system for $\eta = 1$.

Thus, for $\eta\theta < 1$ we obtain a unique optimal feedback control $u_*(x) = u(x, \phi_*(x))$, $v_*(x) = v(x, \phi_*(x))$, $L^A_*(x) = L^A(x, \phi_*(x))$ according to formulas (48), (54) and (49), (53).

Let us summarize the above analysis of the Hamiltonian system as follows:

Theorem 3. (a) *If $\eta = 1$ and $\theta = 1$, then there is a unique optimal control $\tilde{w}(\cdot) = (L^A_*(\cdot), u_*(\cdot), v_*(\cdot))$ in problem* (P1), *with*

$$L^A_*(t) \equiv \max\Big\{0, L - \frac{\alpha\rho}{\varkappa}\Big\}, \qquad u_*(t) \equiv \rho, \qquad v_*(t) \equiv 0 \qquad \text{for all} \quad t \in [0, \infty).$$

In this case $x(t) \equiv x_0 = 1$, $t \ge 0$ is a unique admissible trajectory (*see* (12)).

(b) *If* $\eta\theta < 1$, *then there is a unique optimal feedback control* (*optimal synthesis*) $\tilde{w}_*(x) = (L_*^A(x), u_*(x), v_*(x))$ *in problem* (P1), *with* $L_*^A(x) = L^A(x, \phi_*(x))$, $u_*(x) = u(x, \phi_*(x))$ *and* $v_*(x) = v(x, \phi_*(x))$ *determined by formulas* (49), (53), (48) *and* (54). *Here the feedback* $\phi_*(x)$ *is generated by a unique solution* $(\tilde{x}(\cdot), \tilde{\phi}(\cdot))$ *of the Hamiltonian system* (57) *that reaches* (*or tends to*) *the point* $(\bar{x}, 0)$ *from the right, where* (*see* (56))

$$\bar{x} = \frac{\rho^\eta \alpha^\eta (1-\alpha)^{1-\eta}}{L^\eta \eta^\eta (1-\eta)^{1-\eta} \varkappa}.$$

Namely,

(b.1) *If* $\eta\theta < 1$ *and* $x \le \bar{x}$, *then*

$$L_*^A(x) = 0, \qquad u_*(x) = \rho, \qquad v_*(x) = 0.$$

(b.2) *If* $\eta = 1$, $\theta < 1$ *and* $x > \bar{x}$, *then* (*see* (49), (48) *and* (54))

$$L_*^A(x) = L - \frac{\alpha\rho}{(\varkappa - (1-\theta)\phi_*(x))x}, \qquad u_*(x) = \rho, \qquad v_*(x) = 0.$$

In the case of $\eta = 1$ *and* $\theta < 1$, *for any initial state* $x_0 \le \bar{x}$ *the corresponding optimal trajectory* $x_*(\cdot)$ *is* $x_*(t) \equiv x_0$, $t \ge 0$, *while for any initial state* $x_0 > \bar{x}$ *the corresponding optimal trajectory* $x_*(\cdot)$ *monotonically tends to the point* $\bar{x}$ *from the right as* $t \to \infty$.

(b.3) *If* $\eta < 1$, $\theta \le 1$ *and* $x > \bar{x}$, *then* (*see* (53), (48) *and* (54))

$$L_*^A(x) = L - \frac{\alpha\rho(1-\alpha+(1-\eta)\phi_*(x))^{\frac{1-\eta}{\eta}}}{\eta(1-\eta)^{\frac{1-\eta}{\eta}}(x[\varkappa-(1-\theta)\phi_*(x)])^{\frac{1}{\eta}}},$$

$$u_*(x) = \frac{(1-\alpha)\rho}{1-\alpha+(1-\eta)\phi_*(x)},$$

$$v_*(x) = \frac{L((1-\eta)x[\varkappa-(1-\theta)\phi_*(x)])^{\frac{1}{\eta}}}{(1-\alpha+(1-\eta)\phi_*(x))^{\frac{1}{\eta}}} - \frac{\alpha\rho(1-\eta)}{\eta(1-\alpha+(1-\eta)\phi_*(x))}.$$

In the case of $\eta < 1$ *and* $\theta \le 1$, *for any initial state* $x_0 > 0$, *the corresponding optimal trajectory* $x_*(\cdot)$ *monotonically decreases to* 0 *as* $t \to \infty$.

Finally let us analyze the dynamics of the output $Y(\cdot)$ and the knowledge stock $A(\cdot)$ along the optimal trajectory.

If $\eta = \theta = 1$, then (Theorem 3(a)) the optimal controls are $u(t) \equiv \rho$, $v(t) \equiv 0$ and $L^A(t) \equiv \max\{0, L - \frac{\alpha\rho}{\varkappa}\}$. In the case of $L\varkappa \le \alpha\rho$, we have stagnation of the knowledge stock ($\dot{A}(t) \equiv 0$) and depletion of the output ($Y(t) \to 0$ as $t \to \infty$). For $L\varkappa > \alpha\rho$, the knowledge stock grows exponentially, while the output still depletes to zero for $L\varkappa < \rho(\alpha + \varkappa(1-\alpha))$, is constant for $L\varkappa = \rho(\alpha + \varkappa(1-\alpha))$, and grows exponentially for $L\varkappa > \rho(\alpha + \varkappa(1-\alpha))$.

Let us consider the case $\eta\theta < 1$ in more detail. If $x_0 \le \bar{x}$, then we again have stagnation of the knowledge stock and depletion of the output. If $x_0 > \bar{x}$, then the knowledge stock grows in the beginning, but the growth either terminates at a certain instant ($\eta < 1$) or decelerates ($\eta = 1$), so that the knowledge stock never exceeds a certain level determined by the parameters of the system. The output falls to zero in the long run. However, the following proposition shows that it may grow on some initial time interval.

Theorem 4. *Let $\eta\theta < 1$. Then, for sufficiently large initial values x_0 (i.e., for a relatively large initial stock of the exhaustible resource S_0 and/or for a relatively small initial knowledge stock A_0; see* (15)*), the output $Y(\cdot)$ as a function of t increases on some initial time interval $0 < t < \tau$, $\tau > 0$.*

Proof. For large x_0 the initial part of the optimal trajectory lies in Γ_1 and hence $Y(\cdot)$ is continuously differentiable for the corresponding values of t. Let us show that $\dot{Y}(t) > 0$ on the initial time interval $0 < t < \tau$, $\tau > 0$, of the optimal trajectory. We have

$$\dot{Y}(t) = Y(t)\left[\varkappa\frac{\dot{A}(t)}{A(t)} - \alpha\frac{\dot{L}^A(t)}{L - L^A(t)} + (1-\alpha)\frac{\dot{u}(t)}{u(t)} + (1-\alpha)\frac{\dot{S}(t)}{S(t)}\right]$$

$$= Y(t)\left[\varkappa L^A(t)^{\eta} v(t)^{1-\eta} x(t) - \alpha\frac{\dot{L}^A(t)}{L - L^A(t)} + (1-\alpha)\frac{\dot{u}(t)}{u(t)} - (1-\alpha)(u(t)+v(t))\right] \quad (58)$$

(see (1), (2), (13) and (12)), where $u(t) = u_*(x(t))$, $v(t) = v_*(x(t))$ and $L^A(t) = L^A_*(x(t))$.

Let us show that $\dot{\phi}(t) < 0$ along the optimal trajectory in Γ_1. To see this, note that the curve on which $\dot{\phi}(t) = 0$ in Γ_1 is described by the equation

$$\rho\phi + \frac{\alpha\rho}{\eta} = \frac{L(1-\alpha+(1-\eta)\phi)}{(1-\eta)h(x,\phi)^{\frac{1}{\eta}}} = \frac{L(1-\eta)^{\frac{1-\eta}{\eta}}\left(x[\varkappa-(1-\theta)\phi]\right)^{\frac{1}{\eta}}}{(1-\alpha+(1-\eta)\phi)^{\frac{1-\eta}{\eta}}}. \quad (59)$$

This equation defines x as a monotonically increasing function of ϕ. So any trajectory of our system that intersects this curve at some instant τ (at a point different from $(\bar{x}, 0)$) acquires a positive derivative of the ϕ-coordinate and later enters the set Γ_0 (at a point different from $(\bar{x}, \phi)$). Such a trajectory tends to infinity and so it is not optimal. Hence our optimal trajectory lies in Γ_1 completely below the above curve, and $\dot{\phi}(t) < 0$ on it. This immediately implies that $\dot{u}(t) \ge 0$ in (58) (see (48)).

To estimate the second term in the square brackets in (58), we first denote $\zeta(t) = x(t)[\varkappa - (1-\theta)\phi(t)]$, $\zeta_*(x) = x[\varkappa - (1-\theta)\phi_*(x)]$, and calculate (along the optimal trajectory in Γ_1)

$$\dot{\zeta}(t) = \frac{d}{dt}\big(x(t)\big[\varkappa - (1-\theta)\phi(t)\big]\big) = \dot{x}(t)\big[\varkappa - (1-\theta)\phi(t)\big] - (1-\theta)x(t)\dot{\phi}(t)$$

$$= -(1-\eta)\big[u_*(x(t)) + v_*(x(t))\big]\zeta(t) - (1-\theta)\rho x(t)\phi(t) < 0, \quad (60)$$

because $\zeta(t) > 0$ for $(x(t), \phi(t)) \in \Gamma_1$. Then, after some calculations, we find from (49) for $\eta = 1$, from (53) for $\eta < 1$, and from (44), (60) that

$$-\frac{\dot{L}^A(t)}{L - L^A(t)} = \frac{(1-\eta)^2\dot{\phi}(t)}{\eta\big(1-\alpha+(1-\eta)\phi(t)\big)} - \frac{1}{\eta}\frac{\dot{\zeta}(t)}{\zeta(t)}$$

$$> -\frac{(1-\eta)^2 L^A(t)^\eta v(t)^{1-\eta}\zeta(t)}{\eta\big(1-\alpha+(1-\eta)\phi(t)\big)} + \frac{(1-\theta)\rho x(t)\phi(t)}{\zeta(t)}. \qquad (61)$$

If $\eta = 1$ and $\theta < 1$, then the right-hand side of (61) is positive; hence $\frac{dL_*^A(x)}{dx} > 0$ and

$$L_*^A(x)^\eta v_*(x)^{1-\eta} x \to +\infty \qquad \text{as} \ \ x \to +\infty. \qquad (62)$$

This obviously implies that $\dot{Y}(t) > 0$ for large $x(t)$ along the optimal trajectory, as the second and third terms in the square brackets in (58) are nonnegative, while the last term is bounded due to the restrictions $u(t) \le \rho$ and $v(t) = 0$.

If $\eta < 1$ and $\theta \le 1$, then $\phi_*(x) < \varkappa/(1-\theta)$ in Γ_1. Let us show that $\phi_*(x) \to \varkappa/(1-\theta)$ as $x \to \infty$. Indeed, suppose the contrary. Then it follows from (53) that $L_*^A(x) \to L$ as $x \to \infty$, and due to (52) $v_*(x) \sim x^{1/\eta}$ as $x \to \infty$. Therefore,

$$\frac{d\phi_*(x)}{dx} = \frac{\dot{\phi}(t)}{\dot{x}(t)} = \frac{L_*^A(x)^\eta v_*(x)^{1-\eta} x[\varkappa - (1-\theta)\phi_*(x)] - \rho\phi_*(x)}{(1-\eta)[u_*(x)+v_*(x)]x + (1-\theta)L_*^A(x)^\eta v_*(x)^{1-\eta}x^2} \sim \frac{1}{x}, \qquad (63)$$

which contradicts the boundedness of $\phi_*(\cdot)$. Thus, $\phi_*(x) \to \varkappa/(1-\theta)$ as $x \to \infty$.

If $\zeta_*(\cdot)$ is unbounded, then by (53) $L_*^A(x) \to L$ as $x \to \infty$, and by (52) $v_*(x) \sim \zeta_*(x)^{1/\eta} = o(x^{1/\eta})$ and $v_*(x) \to \infty$ as $x \to \infty$. This shows that the first term in the square brackets in (61) dominates all the negative terms there, and so $\dot{Y}(t) > 0$ for large $x(t)$ along the optimal trajectory.

If $\zeta_*(\cdot)$ is bounded, then $v_*(\cdot)$ is bounded by (51). Hence the right-hand side of (61) is positive for large $x(t)$ and, in particular, $\frac{dL_*^A(x)}{dx} > 0$ for large x. Therefore, again by (51), $v_*(x)$ is bounded away from zero for large x. We see that (62) holds in this case as well, which again implies that $\dot{Y}(t) > 0$ for large $x(t)$ along the optimal trajectory.

Finally, consider the case of $\eta < 1$ and $\theta = 1$. In this case $\zeta(t) = \varkappa x(t)$. Multiplying equation (50) raised to the power η by equation (51) raised to the power $1-\eta$, we find that

$$\eta^\eta(1-\eta)^{1-\eta}\varkappa x = \frac{\alpha^\eta\rho^\eta\big(1-\alpha+(1-\eta)\phi_*(x)\big)^{1-\eta}}{(L - L_*^A(x))^\eta}$$

Recall that $\phi_*(\cdot)$ is a monotonically increasing function of x. If it were bounded, then we would have $L_*^A(x) \to L$ as $x \to \infty$, $v_*(x)^\eta \sim x$ by (51), and hence (63) would be valid, which is impossible for a bounded $\phi_*(\cdot)$. Thus, $\phi_*(x) \to \infty$ as $x \to \infty$.

On the other hand, $\phi_*(x) = O(x)$ because the optimal trajectory lies below the curve described by (59). Therefore, $L_*^A(x) \to L$ as $x \to \infty$ by (53) and $v_*(x) \ge v_0$ for some $v_0 > 0$ and for all sufficiently large x by (54). At the same time, $v_*(x)^\eta = o(x)$ by (54). This shows that the first term in the square brackets in (61) dominates all the negative terms there, and so $\dot{Y}(t) > 0$ for large $x(t)$ along the optimal trajectory.

We showed that for $\eta\theta < 1$ the output $Y(t)$ increases on some initial time interval provided that the initial supply of exhaustible resource S_0 is large and/or the initial knowledge stock A_0 is small. $\square$

6 Discussion

Dynamics of the output $Y(\cdot)$ and the knowledge stock $A(\cdot)$ along the optimal trajectory are depicted in Figures 1 and 2. It follows from the above analysis that optimal growth is only sustainable if the following three conditions hold simultaneously:

(i) the exhaustible resource is not an input to the production of knowledge;

(ii) the accumulation of knowledge has strong scale effects;

(iii) the population is not too small.

In this scenario the growth of output is exponential. The resulting dynamics correspond to the optimal balanced growth path. In this case for a sufficiently large population size L, a constant fraction of labor is allocated to research. The lower the discount rate ρ, the higher this fraction. The fraction also depends on the elasticity of substitution in the production function. The optimal extraction policy implies an exponential depletion of the stock of the exhaustible resource, with the rate equal to the discount rate. This is the well-known Hotelling rule for the optimal depletion of exhaustible resources (see [13]). In sum this implies an exponential growth of the knowledge stock $A(\cdot)$.

Requirement of strong scale effects (ii) for a balanced growth is the opposite to that obtained by Jones [15]. In his model, strong scale effects, coupled with an exponential growth of labor supply, lead to a double-exponential growth of output. This result follows from the assumption of an exponential population growth, which is unrealistic in the long run [22]. Exponential population growth implies constant birth and death rates that are independent of the current population density. Second, more relevant here, exponential growth implies an arbitrarily large population in the long run. This is problematic in view of a finite resource base, a defining feature of our framework.

In the most realistic case $\eta\theta < 1$ we may have two qualitatively different optimal policies depending on whether the accumulation of knowledge requires the resource:

(i) When the accumulation of knowledge is independent of the resource ($\eta = 1$), the fraction of labor employed in research tends from an initially positive value to zero. This means that the research effort becomes successively smaller. The extraction policy is identical to that in the case of optimal sustainable growth described above. The stock of the exhaustible resource depletes exponentially with the rate equal to the discount rate (the Hotelling rule). The policy described above is optimal provided the initial knowledge stock is not too large ($x_0 > \bar{x}$). Otherwise it is optimal to allocate the entire labor to production from the onset.

(ii) When the accumulation of knowledge requires the resource ($\eta < 1$), it is optimal to conduct research until a certain ratio (characterized by (56)) between the knowledge stock and the current supply of the resource is reached. In this case the labor and resource allocated to research gradually decrease and ultimately vanish at the moment of reaching the above-mentioned ratio. Afterwards the research effort stops and the stock of knowledge remains at its maximum level. This policy is optimal when $x_0 > \bar{x}$. For $x_0 \leq \bar{x}$ it is optimal not to invest in research as the initial knowledge stock is sufficiently large; the optimal extraction policy follows the Hotelling rule in this case.

Finally, condition (iii) says that a sufficiently small economy (with $L\varkappa \leq \alpha\rho$) will not grow, even under strong scale effects and even if the accumulation of knowledge does not depend on the exhaustible resource. This minimum size condition is the least restrictive of all conditions and can be assumed to hold a priori. In the typical case $\varkappa = 1$, we have $L > \alpha\rho$. This inequality can be maintained in all cases of interest since L is the size of the labor force, $\alpha < 1$ and ρ is the discount rate. The case $L\varkappa \leq \alpha\rho$ is included for completeness.

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