OPTIMAL STATIONARY SOLUTION IN FOREST MANAGEMENT MODEL BY ACCOUNTING INTRA-SPECIES COMPETITION

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1. Introduction

We consider forest management model described by equation

$$\frac{\partial x(t,l)}{\partial t} + \frac{\partial\big[g(l,E(t))x(t,l)\big]}{\partial l} = -[\mu(l,E(t)) + u(l)]x(t,l), \tag{1}$$

where $x(t,l)$ is the density of the trees of size l at the moment t, E characterizes intra-species competition and has the form

$$E(t) = \chi \int_0^L l^2 x(t,l)dl \tag{2}$$

with some positive $\chi > 0$, g and μ are respectively the growth and mortality rates of these trees, and $u(l)$ accounts the exploitation intensity of the forest. Such form of the growth and mortality rates was used in [2]. We assume these rates are continuous function being separated from zero on the interval $[0,L], L>0$, of sizes, where we manage and exploit the forest, for all reasonable values of the second argument, for example, for all E bounded by sufficiently big positive constant M.

The reforestation is defined by the boundary condition which is the sum of the natural seeding and the density p of planted trees

$$x(t,0) = \int_0^L r(l)x(t,l)dl + p(t) \tag{3}$$

where r is the reproduction coefficient. It is naturally to assume that the reproduction coefficient r is a nonnegative function being positive near the right end of the interval $[0,L]$.

In this paper, at the first, we prove that by a selected intensity of exploitation and constant positive planting $p(t) \equiv p_0 > 0$ there exists nontrivial stationary solution $x = x(l,E)$ in the model (1), (2), (3) under the following assumptions on the growth and mortality rates

$$g(.,E_1) > g(.,E_2), \qquad \mu(.,E_1) < \mu(.,E_2), \qquad \frac{g(0,E_1)}{g(l,E_1)} > \frac{g(0,E_2)}{g(l,E_2)}, \tag{4}$$

when $E_1 < E_2$, which looks as reasonable. Indeed, the conditions (4) characterize respectively the decreasing of growth rate and the increasing of the mortality, and else more significant influence on the growth rate of smaller size individuum under the increasing of the exponent E.

And at the second we prove the existence of stationary solution providing the maximum profit of exploitation.

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2. Existence of stationary solution

Theorem 1. *For a selected measurable intensity of exploitation u and constant positive planting $p(t) \equiv p_0 \geq 0$ in problem (1), (2), (3) with continuous g, μ and r there exists unique stationary solution under assumptions (4), if*

$$\int_0^L r(l)\frac{g(0,E)}{g(l,E)} e^{-\int_0^l m(s,E)ds} dl < 1 \tag{5}$$

for all values of exponent E in its range.

Really, such a stationary solution $x, x = x(l,E)$, has to satisfy equation (1) in form

$$\frac{d\big[g(l,E)x(l,E)\big]}{dl} = -[\mu(l,E) + u(l)]x(l,E), \tag{6}$$

where E is the respective constant value of the exponent (2). The value E depends on the solution x but in the beginning we consider it as independent parameter. In such a case the solution of the last equation could be easily found. It has the form

$$x(l,E) = \frac{g(0,E)x(0,E)}{g(l,E)} e^{-\int_0^l m(s,E)ds} \quad \text{where} \quad m(s,E) = \frac{\mu(s,E) + u(s)}{g(l,E)} \tag{7}$$

Substituting this expression in (3) we get the value $x_0 := x(0,E)$:

$$x_0 = \frac{p_0}{1 - \int_0^L r(l)\frac{g(0,E)}{g(l,E)} e^{-\int_0^l m(s,E)ds} dl} \tag{8}$$

Due to condition (5) x_0 is positive and finite. Hence, the stationary solution is

$$x(l,E) = \frac{x_0 g(0,E)}{g(l,E)} e^{-\int_0^l m(s,E)ds} \tag{9}$$

Proposition 2.1. *For any $l \in [0, L]$ the solution (9) is decreasing function on the parameter $E \geq 0$ by assumptions (4).*

Indeed the integrant of the main integral in (8) is decreasing function on E due to the assumptions, and so x_0 decreases by increasing of E. But the decreasing of x_0 and these assumptions imply immediately the decreasing of $x(l, .)$ due to formula (9). Consequently Proposition 2.1 is true.

Now substituting of stationary solution (9) into integral in (2) we get continuous positive function f on the parameter E :

$$f(E) = \chi \int_0^L l^2 x(l,E)dl$$

Proposition 2.1 implies

Corollary 2.1. *By assumptions (4) the function f is decreasing function on parameter $E \geq 0$.*

Consequently by the increasing of the parameter E from zero up to $f(0)$ the value of this function decreases from $f(0)$ up to smaller value $f(f(0))$. Hence the difference $e - f(e)$ is increasing function on the interval $[0, f(0)]$ and has values of different signs at the ends of this interval. Consequently there exists only one value $E_0 \in [0, f(0)]$ at which this difference has zero value. But that means that for the solution $x(., E_0)$ the respective value of competition parameter E is E_0, and so $x(., E_0)$ is needed stationary solution.

Now, for a given $E = E_0$ and the new variable $z := g(l, E_0)x$ the right hand side of equation (6) satisfies Lypschitz condition with respect to z. Hence due to uniqueness theorem for the Cauchy problem [1] there exists only one solution of this equation satisfying condition $z(0) = g(0, E_0)x_0$

Theorem 1 is proved.

Remark 1. In [4] the existence of stationary solution being steady state was established for Lipschitz growth rate and mortality rates in the case without exploitation, that is $u \equiv 0$.

3. Existence of stationary solution with maximum profit

The selection of a better exploitation of the population could be considered as optimization problem

$$\int_0^L c(l)u(l)x(l,E)dl + c_L x(L,E) - p_0 c_0 \to \max \tag{10}$$

where c, c_L, c_0 are the respective aggregated prices; constant value p_0 and a measurable function u,

$$0 \leq p_0 \leq P_0, \qquad 0 \leq u_1(l) \leq u \leq u_2(l), \tag{11}$$

are variables to be selected to maximize the profit. Here a positive constant P_0 and piecewise continuous functions u_1 and u_2 characterize technological or ecological constraints.

The substitution of stationary solution (9) into the functional in (10) leads to form

$$x_0 g(0,E) \left[\int\limits_0^L c(l) e^{-\int\limits_0^l \frac{\mu(s,E)}{g(s,E)} dl - \phi(l,E)} d\phi(l,E) + \frac{c_L e^{-\int\limits_0^L \frac{\mu(s,E)}{g(s,E)} dl - \phi(L,E)}}{g(L,E)} \right] - p_0 c_0 \tag{12}$$

of this functional, where

$$\phi(l,E) = \int\limits_0^l \frac{u(s)}{g(s,E)} ds. \tag{13}$$

Thus one needs to find maximum of functional (12) with respect to p_0 and u under constraints (11).

Theorem 2. *There exists a stationary solution maximizing the profit (12) if μ, g, and c are continuous functions on variables l and E, and else the positive functions g and μ satisfies conditions (4) and are separated from zero.*

The following statement is useful.

Lemma 3.1. *By conditions of Theorem 2 profit (12) is bounded functional on the space of pairs of a value p_0 and a measurable function u, which satisfy constraints (11).*

Indeed, we have

$$\left| x_0 g(0,E) \left[\int_0^L c(l) e^{-\int_0^l \frac{\mu(s,E)}{g(s,E)} dl - \phi(l,E)} d\phi(l,E) + \frac{c_L e^{-\int_0^L \frac{\mu(s,E)}{g(s,E)} dl - \phi(L,E)}}{g(L,E)} \right] - p_0 c_0 \right| \leq$$

$$\leq x_0 g(0,E) \left(\left| \int_0^L c(l) e^{-\phi(l,E)} d\phi(l,E) \right| + \frac{c_L}{g(L,f(0))} \right) + c_0 P_0 \leq$$

$$\leq x_0 g(0,0) \left(C + c_L \frac{1}{g(L,f(0))} \right) + c_0 P_0 < \infty,$$

where $g(0,0)$, $g(L,f(0))$ and P_0 are finite due to our assumptions, $C = \sup\limits_{l \in [0,L]} c(l)$ is finite since in the strength of continuity of c, and finally x_0 is clearly bounded. Consequently, the profit functional is bounded.

Consider now the least upper bound of possible values of this functional and a sequence of pairs of controls u_k and values p_k satisfying conditions (11) such that the respective value of functional converges to this bound when $k \to \infty$. Denote by E_k the respective values of competition parameter.

Due to that all p_k are in interval $[0, P_0]$ there exists convergent subsequence p_{k_n} by $k_n \to \infty$. Without loss of generality we count that $p_k \to p_\infty$ when $k \to \infty$.

All possible values of competition parameter are bounded by $f(0)$, and so there exists subsequence $E_{k_n} \to E_\infty$ by $k_n \to \infty$. An analogously we count $E_k \to E_\infty$ by $k \to \infty$.

Thus we arrive to sequence of triplets $\{u_k, p_k, E_k\}$ with second and the third components having finite limits by $k \to \infty$.

For the control u_k and any $l_1, l_2 \in [0, L], l_1 \leq l_2$, the respective sequence ϕ_k satisfies inequalities

$$\int_{l_1}^{l_2} \frac{u_1(l)}{g(l,E_k)} dl \leq \phi_k(l_2) - \phi_k(l_1) \leq \int_{l_1}^{l_2} \frac{u_2(l)}{g(l,E_k)} dl. \tag{14}$$

as it is easy to see. In particular, all ϕ_k satisfy the Lypschitz condition with constant being the maximum of the function $u_2(\cdot)/g(\cdot, f(0))$ on the interval $[0, L]$. Consequently, the set of functions ϕ_k is bounded and equicontinuous on this interval. Hence due to the Arzela-Ascoli theorem [3] there exists a subsequence $\{\phi_{k_n}\}$ that uniformly converges to some function ϕ_∞ as $k_n \to \infty$.

The profit functional (12) continuously depends on ϕ, p_0 and E, as it is easy to see. Hence this functional attains its maximum value by $\phi = \phi_\infty, p_0 = p_\infty$ and $E = E_\infty$.

To finish the proof one needs to find an admissible control u_∞ which gives the limit function ϕ_∞ by formula (13). But, as it is easy to see, this function also satisfies inequalities (14) and is absolutely continuous. Hence its derivative exists almost everywhere in $[0, L]$ and satisfies the inequality

$$\frac{u_1(l)}{g(l, E_\infty)} \leq \phi'(l) \leq \frac{u_2(l)}{g(l, E_\infty)}$$

at each point of its existence. Consequently, one can define the control u_∞ by the formula $u(l) = g(l, E_\infty)\phi'_\infty(l)$ at any such a point and take any value in $[u_1, u_2]$ for it in the rest part of the interval $[0, L]$.

Theorem 2 is proved.

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