Optimization of Trends in Resource Productivity for Providing Sustainable Economic Development

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Abstract

In this paper, a dynamic optimization model of investment in improvement of the resource productivity index is analyzed for obtaining balanced economic growth trends including both the consumption index and natural resources use. The research is closely connected with the problem of shortages of natural resources stocks, the security of supply of energy and materials, and the environmental effectiveness of their consumption. The main idea of the model is to introduce an integrated environment for elaboration of a control policy for management of the investment process in development of basic production factors such as capital, energy and material consumption. An essential feature of the model is providing the possibility to invest in economy's dematerialization. Another important construction is connected with the price formation mechanism which presumes the rapid growth of prices on exhausting materials. The balance is formed in the consumption index which negatively depends on growing prices on materials. The optimal control problem for the investment process is posed and solved within the Pontryagin maximum principle. Specifically, the growth and decline trends of the Hamiltonian trajectories are examined for the optimal solution. It is proved that for specific range of the model parameters there exists the unique steady state of the Hamiltonian system. The steady state can be interpreted as the optimal steady trajectory along which investments in improving resource productivity provide raising resource efficiency and balancing this trend with growth of the consumption index. The comparison analysis is implemented for optimal model trends and historical trends of real

econometric data. As a result of system analysis and modeling, one can elaborate investment strategies in economy's dematerialization, resource and environmental management for improving the resource productivity index and, consequently, for shifting the economic system from nonoptimal paths to the trajectory of sustainable development.

Introduction

The paper is devoted to the problem of optimizing trends in resources productivity and balancing investment in economy's dematerialization with sustainable growth of the consumption index. The problem is considered within the classical approach [Solow, 1970], [Schell, 1969] of construction of economic growth models. The main new element in the proposed model is a price formation mechanism which reflects possibility of rapid growth of prices on exhausting resources. Growing prices negatively influence on the consumption index which should be maximized in the model as the basic element of the utility function.

Let us note that the stated problem has in its background very important concerns of the modern society with respect to the current world resource utilization. The recent statistics [IPCC Report, 2007] [OECD Report, 2008] shows rapid increase of natural resource consumption, especially, in the following components: fossil energy (oil, natural gas, oil), ferrous metals (iron ore, etc.), nonferrous metals (bauxite, etc.), non-metalliferous minerals (lime), biomass (wood, etc.). Taking into account the limitations of natural resources, at least, of its assured part, the problem of raising resource efficiency and even reducing resource consumption becomes extremely significant. Nowadays, a comprehensive research is being implemented on material flow analysis (MFA) by international (EUROSTAT, IPCC, OECD, World Resources Institute) and national (Germany, the Netherlands, the United States, Japan, China) research and policy making organizations. Material flow analysis is a systematic assessment of the flows and stocks of materials within a system defined in space and time. It connects the sources, the pathways, and the intermediate and the final sinks of a material. The method is an attractive decision-support tool in resource management, waste management, and environmental management.

In this paper, we supplement this research and develop the model of dynamic optimization of investment process in improving resource productivity within the economic growth theory [Arrow, 1985], [Ayres, Warr, 2009], [Barro, Sala-i-Martin, 1995], [Crespo-Cuaresma, Palokangas, Tarasyev, 2011], [Gordon, Koopmans, Nordhaus, Skinner, 1988], [Grossman, Helpman, 1991]. Particularly, the construction of the model inherits elements of economic growth models introduced in [Ane, Watanabe, Tarasyev, 2007a, 2007b], [Ayres, Krasovskii, Tarasyev, 2009], [Krasovskii, A.A., Tarasyev, 2009], [Krasovskii, Kryazhimskiy, Tarasyev, 2008], [Sanderson, Tarasyev, Usova, 2010], [Tarasyev, Watanabe, 2001], [Tarasyev, Watanabe, Zhu, 2002], [Watanabe, Shin, Heikkinen, Tarasyev, 2009]. Let us mention here papers [Aseev, Besov, Kaniovski, 2011], [Feichtinger, Hartl, Kort, Veliov, 2006], [Hutschenreiter, Kaniovski, Kryazhimskii, 1995] which are devoted to different aspects of economic growth modeling and conceptually are close to our approach. The model dynamics includes production, current material use and cumulative material consumption as main phase variables. Growing trend in production is given exogenously by the exponential term generated by such production factors as capital and labor. Material use is introduced as a production factor in the production function of the Cobb-Douglas type. The main control variable is presented by investment in raising resource productivity in the current period.

It is assumed that prices on materials due to exhaustion are growing rapidly to infinity when the cumulative material consumption is close to the available (assured) stock. In the balance equation both growth and decline trends are taken into account: the growth trend in the consumption index is stimulated by the production growth and the decline trend is caused by raising costs of materials and expenditures directed on improvement of resource productivity.

The problem is to find the optimal proportion of investment in the dynamic process with maximization of the utility function given as the integrated consumption index over trajectories of the economic system. The model is examined within the framework of the Pontryagin maximum principle [Pontryagin, Boltyanskii, Gamkrelidze, Mishchenko, 1962] with special characteristics of infinite horizon [Aseev, Kryazhimskiy, 2007]. Specific features of the corresponding Hamiltonian system are examined within the qualitative theory of differential equations [Hartman, 1964]. In our analysis we use constructions of dynamic programming and the theory of generalized solutions of Hamilton-Jacobi equations [Bellman, 1957], [Krasovskii, A.N., Krasovskii, N.N., 1995], [Subbotin, 1995], [Rockafellar, 2004]. The range of model parameters is indicated for existence and uniqueness of a steady state. The steady state plays the role of the optimal steady solution and its proportions can be used as an economic standard for the first approximation of solution of the optimal control problem. It is shown that at the steady state the optimal level of investment in resource productivity provides reduce in resource consumption and raise of its efficiency, and establish a reasonable balance between investment and consumption.

We provide the comparison analysis and adjustment of optimal model trajectories to historical trends of real econometric data. This analysis shows that the model quite adequately catches the main econometric tendencies and reflects the influence of investment in improvement of resource productivity on sustainable growth under limited resources. The main output of the implemented analysis and modeling is construction of investment strategies in economy's dematerialization and improvement of resource productivity. The model simulations demonstrate that the proposed investment strategies could shift the economic system from nonoptimal paths to the trajectory of sustainable development.

The Model Description

In this section we introduce the main variables and model parameters and provide the basic relations for the model construction.

Model Variables

We assume that the model dynamics is evolved in time t on the infinite horizon $t \in [0, +\infty]$.

The main phase variables of the model are presented by the current production, the resource use and the cumulative resource consumption.

The symbol $y = y(t)$ stands for production in period t.

The resource use in period t is denoted by the symbol $m = m(t)$ with the initial condition $m(0) = m^*$.

The cumulative resource consumption is introduced as the integrated material use

$$M = M(t) = \int_0^t m(s)\,ds\,. \tag{1}$$

The initial condition for the cumulative resource consumption is given by the relation $M(0) = M^* = 0$.

By the symbol $z = z(t)$ we denote the resource productivity in period t:

$$z(t) = \frac{y(t)}{m(t)}\,. \tag{2}$$

Sometimes for convenience we use the value of resource intensity inverse to productivity

$$Z(t) = \frac{1}{z(t)} = \frac{m(t)}{y(t)}\,. \tag{3}$$

Rates of the main variables are introduced in the usual way:

the symbol $\frac{dy(t)}{dt}$ stands for the rate of production in period t;

the symbol $\frac{dm(t)}{dt}$ is introduced for the rate of the material use;

the meaning of the symbol $\frac{dM(t)}{dt} = m(t)$ is the rate the cumulative material consumption;

by the symbol $\frac{dz(t)}{dt}$ we denote the rate of the material productivity in period t.

Price Formation Mechanism

In the definition of the price formation mechanism the basis is provided by the concept of raising prices $p(t)$ on natural resources in the case of their limitation or exhaustion. It is assumed that prices are growing according to the inversely proportional rule of resources exhaustion

$$p = p(t) = p_0 \frac{1}{\left(1 - \frac{M(t)}{M_0}\right)^{\gamma}} \tag{4}$$

Here the meter $\gamma, \gamma \geq 0$, is the elasticity coefficient of the price formation mechanism, the symbol M_0 stands for the limitation of natural resources, and the symbol p_0 denotes the initial price on natural resources.

Balance Equation

In the balance equation it is taken into account that production $y(t)$ in period t is shared between consumption $c(t)$, from the one hand, and the growing cost of natural resources $p(t)\,m(t)$ plus investment $s(t)$ in improving the resource productivity, from the other hand,

$$y(t) = c(t) + p(t)\,m(t) + s(t)\,. \qquad (5)$$

Let us assume that there exists an upper bound s^0 for investment $s(t)$

$$0 \leq s(t) \leq s^0 < y(t)\,.$$

Deducing the consumption intensity $c(t)/y(t)$ from this equation through the resource intensity $m(t)/y(t)$ we obtain the following relation

$$\frac{c(t)}{y(t)} = 1 - p(t)\frac{m(t)}{y(t)} - u(t)\,. \qquad (6)$$

Here the symbol $u(t)$ stands for the investment intensity

$$u(t) = \frac{s(t)}{y(t)}\,. \qquad (7)$$

We assume that there exists an upper bound u^0 for the investment intensity $u(t)$, so $u(t) \leq u^0$.

Production Function

The exponential production function of the Cobb-Douglas type is selected for the first version of the model

$$y(t) = a\,e^{bt}\,m^{\alpha}(t)\,. \qquad (8)$$

Here the parameter $a, a > 0$, is a scale factor; the growth rate $b, b \geq 0$, indicates the growth process of production $y(t)$ due to development of basic production factors such as capital, labor, technology, etc.; the symbol $\alpha, \alpha \geq 0$, denotes the elasticity coefficient of natural resources. We assume the diminishing return to scale of natural resources as a production factor, $0 \leq \alpha < 1$.

Consumption

Let us obtain formula for the consumption intensity expressed through the resources consumption $m(t), M(t)$, by substituting relations of the price formation mechanism (4) and the production function (8) to relation (6)

$$\frac{c(t)}{y(t)} = 1 - \frac{p_0}{a\,e^{bt}\left(1 - \frac{M(t)}{M_0}\right)^{\gamma}} m^{(1-\alpha)}(t) - u(t)\,. \qquad (9)$$

Model Dynamics

Let assume that the relative raise in the resource productivity $z(t)$ is proportional to the portion of the assigned investment $u(t)$

$$\frac{1}{z(t)}\frac{dz(t)}{dt} = \beta u(t)\,. \qquad (10)$$

Here the parameter $\beta, \beta \geq 0$, describes the effectiveness of investments investment $u(t)$ in raising the resource productivity.

Taking into account the definition (2) of the resource productivity $z(t)$ one can obtain the following presentation for its rate

$$\frac{1}{z(t)}\frac{dz(t)}{dt} = \frac{dy(t)}{y(t)} - \frac{dm(t)}{m(t)}\,. \qquad (11)$$

The last equation means that the rate of the resource productivity can be decomposed into two components: the production rate and the rate of the resource consumption.

Developing this formula further on the basis of the presentation for the production function (8) we get the following relation

$$\frac{1}{z(t)}\frac{dz(t)}{dt} = b - (1-\alpha)\frac{dm(t)}{m(t)}\,. \qquad (12)$$

Finally, combining formulas (10) and (12) we derive the equation for the rate of the resource consumption

$$\frac{dm(t)}{m(t)} = \frac{1}{(1-\alpha)}\left(b - \beta u(t)\right). \qquad (13)$$

Equation (13) shows that the rate of the resource consumption is influenced by the production growth rate b and can be reduced only by investment $u(t)$ in raising the resource productivity. Let us note that if investment is equal to zero, $u(t)=0$, then the rate of the resource consumption should be proportional to the production growth rate b.

To develop the model dynamics further, let us introduce the following change of variables

$$x_1(t) = e^{\frac{-b\gamma t}{(1-\alpha-\gamma)}} \left(1 - \frac{M(t)}{M_0}\right)^{\gamma}, \tag{14}$$

$$x_2(t) = e^{\frac{-bt}{(1-\alpha-\gamma)}} m(t). \tag{15}$$

We derive the differential equations for the model dynamics by differentiating variables $x_1(t)$, $x_2(t)$ in time t and taking into account equations (1)-(2), (8), (13). We obtain the following differential equations which form the basic model dynamics

$$\frac{dx_1(t)}{dt} = -\frac{b\gamma}{(1-\alpha-\gamma)} x_1(t) - \frac{\gamma}{M_0} x_1^{\left(1-\frac{1}{\gamma}\right)}(t)\, x_2(t), \tag{16}$$

$$\frac{dx_2(t)}{dt} = \frac{1}{(1-\alpha)} \left(-\frac{b\gamma}{(1-\alpha-\gamma)} - \beta u(t)\right) x_2(t). \tag{17}$$

The initial conditions are given by the following relations

$$x_1(0) = \left(1 - \frac{M(0)}{M_0}\right)^{\gamma} = \left(1 - \frac{M^*}{M_0}\right)^{\gamma} = 1, \tag{18}$$

$$x_2(0) = m(0) = m^*. \tag{19}$$

Relations (18), (19) mean the variable $x_1(t)$ is an analogue of the cumulative resource consumption $M(t)$, and the variable $x_2(t)$ is equivalent to the current resource use $m(t)$.

It is important to remind that the control variable $u(t)$ in the model dynamics (16)-(17) is subject to constraints

$$0 \le u(t) \le u^0 < 1. \tag{20}$$

Logarithmic Consumption Index

Using variables $x_1(t)$, $x_2(t)$ we introduce the logarithmic consumption index in period t

$$\ln c(t) = \ln y(t) + \ln\left(1 - \frac{p_0}{a}\frac{x_2^{(1-\alpha)}(t)}{x_1(t)} - u(t)\right) =$$

$$= \ln\left(a e^{bt} m^{\alpha}(t)\right) + \ln\left(1 - \frac{p_0}{a}\frac{x_2^{(1-\alpha)}(t)}{x_1(t)} - u(t)\right) =$$

$$= \ln\left(a e^{\frac{(1-\gamma)bt}{(1-\alpha-\gamma)}}\right) + \alpha \ln x_2(t) + \ln\left(1 - \frac{p_0}{a}\frac{x_2^{(1-\alpha)}(t)}{x_1(t)} - u(t)\right). \qquad (21)$$

Considering the model dynamics (14)-(15) on the time horizon $[0,T), T \le +\infty,$ we introduce the integrated logarithmic index discounted with the discount rate $\rho, \rho > 0,$

$$J\left(x_1(\cdot), x_2(\cdot), u(\cdot)\right) = ,$$

$$= \int_0^T e^{-\rho t}\left(\ln a + \frac{(1-\gamma)bt}{(1-\alpha-\gamma)} + \alpha \ln x_2(t) + \ln\left(1 - \frac{p_0}{a}\frac{x_2^{(1-\alpha)}(t)}{x_1(t)} - u(t)\right)\right)dt, \qquad (22)$$

as the utility function for the optimal control problem.

Optimal Control Problem

We pose the optimal control problem related to the goal of raising the resource productivity. Namely, the problem is to maximize the utility function (22) over control processes $\left(x_1(t), x_2(t), u(t)\right)$ of the dynamic system (16)-((17) satisfying the initial conditions (18)-(19) and subject to constraints (20) for the control parameter $u(t)$.

Special Case $\gamma = 1$

Let us consider the special case when the elasticity coefficient in the price formation mechanism has the unit value, $\gamma = 1$.

In this case the phase variables have the following form

$$x_1(t) = e^{\frac{bt}{\alpha}}\left(1 - \frac{M(t)}{M_0}\right), \qquad (23)$$

$$x_2(t) = e^{\frac{bt}{\alpha}}\ m(t)\,. \tag{24}$$

The model dynamics is described by the system of differential equations

$$\frac{dx_1(t)}{dt} = \frac{b}{\alpha}\ x_1(t) - \frac{1}{M_0}\, x_2(t)\,, \tag{25}$$

$$\frac{dx_2(t)}{dt} = \frac{1}{(1-\alpha)}\left(\frac{b}{\alpha} - \beta u(t)\right) x_2(t)\,, \tag{26}$$

with initial conditions

$$x_1(0) = 1\,, \tag{27}$$

$$x_2(0) = m^*\,, \tag{28}$$

and constraints

$$0 \le u(t) \le u^0 < 1 \tag{29}$$

for control parameter $u(t)$.

The utility function has the form similar to the structure (22)

$$J\left(x_1(\cdot), x_2(\cdot), u(\cdot)\right) = \int_0^T e^{-\rho t}\left(\alpha \ln x_2(t) + \ln\left(1 - \frac{p_0}{a}\frac{x_2^{(1-\alpha)}(t)}{x_1(t)} - u(t)\right)\right) dt\,. \tag{30}$$

The Hamiltonian of the Optimal Control Problem

Let us introduce the Hamiltonian function for the optimal control problem (25)-(30)

$$\tilde{H}\left(x_1, x_2, u, t, \tilde{\psi}_1, \tilde{\psi}_2\right) = e^{-\rho t}\left(\alpha \ln x_2 + \ln\left(1 - \frac{p_0}{a}\frac{x_2^{(1-\alpha)}}{x_1} - u\right)\right) +$$

$$+\tilde{\psi}_1 \frac{b}{\alpha} x_1 - \tilde{\psi}_1 \frac{1}{M_0} x_2 + \tilde{\psi}_2 \frac{1}{(1-\alpha)} x_2 \left(\frac{b}{\alpha} - \beta u\right). \tag{31}$$

Here parameters $\tilde{\psi}_1,\ \tilde{\psi}_2$ are adjoint variables for the phase variables $x_1,\ x_2$.

Implementing the following change of variables

$$\psi_1(t) = e^{\rho t}\, \tilde{\psi}_1(t)\,, \qquad \psi_2(t) = e^{\rho t}\, \tilde{\psi}_2(t)\,, \tag{32}$$

we obtain the expression for the stationary Hamiltonian

$$H\left(x_1, x_2, u, \psi_1, \psi_2\right) = \alpha \ln x_2 + \ln\left(1 - \frac{p_0}{a}\frac{x_2^{(1-\alpha)}}{x_1} - u\right) +$$

$$+\psi_1 \frac{b}{\alpha} x_1 - \psi_1 \frac{1}{M_0} x_2 + \psi_2 \frac{1}{(1-\alpha)} x_2 \left(\frac{b}{\alpha} - \beta u \right), \tag{33}$$

and relations between the Hamiltonians $\tilde{H}$ and H

$$\tilde{H}\left(x_1, x_2, u, t, \tilde{\psi}_1, \tilde{\psi}_2\right) = e^{-\rho t} H\left(x_1, x_2, u, \psi_1, \psi_2\right). \tag{34}$$

The Maximized Hamiltonian

Let us maximize the stationary Hamiltonian H (33) with respect to the control parameter u. It is not difficult to show that the Hamiltonian H is strictly concave with respect to this parameter. Therefore (see [Krasovskii, Tarasyev, 2008]), three maximum regimes for the control parameter u, and, respectively, for the maximized Hamiltonian may take place.

The first regime corresponds to the zero value of the control parameter, $u = 0$. For this regime the maximized Hamiltonian has the following form

$$H_1\left(x_1, x_2, \psi_1, \psi_2\right) = \alpha \ln x_2 + \ln\left(1 - \frac{p_0}{a} \frac{x_2^{(1-\alpha)}}{x_1}\right) +$$

$$+\psi_1 \frac{b}{\alpha} x_1 - \psi_1 \frac{1}{M_0} x_2 + \psi_2 \frac{1}{(1-\alpha)} x_2 \frac{b}{\alpha}. \tag{35}$$

The second regime arises at the upper bound for the control parameter, $u = u^0$. The maximized Hamiltonian in this case is presented by the relation

$$H_2\left(x_1, x_2, \psi_1, \psi_2\right) = \alpha \ln x_2 + \ln\left(1 - \frac{p_0}{a} \frac{x_2^{(1-\alpha)}}{x_1} - u^0\right) +$$

$$+\psi_1 \frac{b}{\alpha} x_1 - \psi_1 \frac{1}{M_0} x_2 + \psi_2 \frac{1}{(1-\alpha)} x_2 \left(\frac{b}{\alpha} - \beta u^0 \right). \tag{36}$$

The third regime is connected with an intermediate maximum value of the optimal parameter and is determined by the maximum condition

$$\frac{\partial H}{\partial u}\left(x_1, x_2, u, \psi_1, \psi_2\right) = -\frac{1}{\left(1 - \frac{p_0}{a} \frac{x_2^{(1-\alpha)}}{x_1} - u\right)} - \frac{\beta \psi_2 x_2}{(1-\alpha)} = 0. \tag{37}$$

Resolving the maximum condition (37) with respect to the control parameter u we obtain the relation for the intermediate maximum value

$$u^* = 1 - \frac{p_0}{a}\frac{x_2^{(1-\alpha)}}{x_1} + \frac{(1-\alpha)}{\beta\psi_2 x_2}. \tag{38}$$

It is clear from formula (37) that the adjoint variable ψ_2 is negative, $\psi_2 < 0$, for the intermediate regime.

The maximized Hamiltonian for the intermediate regime has the following form

$$H_3\left(x_1, x_2, \psi_1, \psi_2\right) = \alpha \ln x_2 + \ln\left(-\frac{(1-\alpha)}{\beta\psi_2 x_2}\right) +$$

$$+\psi_1 \frac{b}{\alpha} x_1 - \psi_1 \frac{1}{M_0} x_2 + \psi_2 \frac{1}{(1-\alpha)} x_2 \left(\frac{b}{\alpha} - \beta + \frac{\beta\, p_0}{a}\frac{x_2^{(1-\alpha)}}{x_1}\right) - 1. \tag{39}$$

The Hamiltonian Systems

Let us compile the Hamiltonian systems for the obtained three control regimes, $i = 1, 2, 3,$ basing on the general constructions of the Pontryagin maximum principle

$$\frac{dx_1(t)}{dt} = \frac{\partial H_i}{\partial \psi_1}\left(x_1(t), x_2(t), \psi_1(t), \psi_2(t)\right),$$

$$\frac{dx_2(t)}{dt} = \frac{\partial H_i}{\partial \psi_2}\left(x_1(t), x_2(t), \psi_1(t), \psi_2(t)\right),$$

$$\frac{d\psi_1(t)}{dt} = \rho\psi_1(t) - \frac{\partial H_i}{\partial x_1}\left(x_1(t), x_2(t), \psi_1(t), \psi_2(t)\right),$$

$$\frac{d\psi_2(t)}{dt} = \rho\psi_2(t) - \frac{\partial H_i}{\partial x_2}\left(x_1(t), x_2(t), \psi_1(t), \psi_2(t)\right). \tag{40}$$

In the first case $(i = 1)$ with the zero control regime, we obtain the following Hamiltonian system

$$\frac{dx_1(t)}{dt} = \frac{b}{\alpha} x_1(t) - \frac{1}{M_0} x_2(t),$$

$$\frac{dx_2(t)}{dt} = \frac{b}{\alpha(1-\alpha)} x_2(t),$$

$$\frac{d\psi_1(t)}{dt} = \rho\psi_1(t) - \frac{b}{\alpha}\psi_1(t) - \frac{p_0}{a}\frac{1}{\left(1 - \frac{p_0}{a}\frac{x_2^{(1-\alpha)}(t)}{x_1(t)}\right)}\frac{x_2^{(1-\alpha)}(t)}{x_1^2(t)},$$

$$\frac{d\psi_2(t)}{dt} = \rho\psi_2(t) - \frac{b}{\alpha(1-\alpha)}\psi_2(t) + \frac{1}{M_0}\psi_1(t) -$$

$$-\frac{\alpha}{x_2(t)} + \frac{p_0\,(1-\alpha)}{a}\frac{1}{\left(1-\frac{p_0}{a}\frac{x_2^{(1-\alpha)}(t)}{x_1(t)}\right)}\frac{x_2^{-\alpha}(t)}{x_1(t)}. \tag{41}$$

In the second case $(i=2)$ for the upper bound control regime, one can get the Hamiltonian system

$$\frac{dx_1(t)}{dt} = \frac{b}{\alpha}x_1(t) - \frac{1}{M_0}x_2(t),$$

$$\frac{dx_2(t)}{dt} = \frac{1}{(1-\alpha)}x_2(t)\left(\frac{b}{\alpha} - \beta u^0\right),$$

$$\frac{d\psi_1(t)}{dt} = \rho\psi_1(t) - \frac{b}{\alpha}\psi_1(t) - \frac{p_0}{a}\frac{1}{\left(1-\frac{p_0}{a}\frac{x_2^{(1-\alpha)}(t)}{x_1(t)} - u^0\right)}\frac{x_2^{(1-\alpha)}(t)}{x_1^2(t)},$$

$$\frac{d\psi_2(t)}{dt} = \rho\psi_2(t) - \frac{1}{(1-\alpha)}\psi_2(t)\left(\frac{b}{\alpha} - \beta u^0\right) + \frac{1}{M_0}\psi_1(t) -$$

$$-\frac{\alpha}{x_2(t)} + \frac{p_0\,(1-\alpha)}{a}\frac{1}{\left(1-\frac{p_0}{a}\frac{x_2^{(1-\alpha)}(t)}{x_1(t)} - u^0\right)}\frac{x_2^{-\alpha}(t)}{x_1(t)}. \tag{42}$$

In the third case $(i=2)$, we have the following Hamiltonian system for the intermediate optimal control regime

$$\frac{dx_1(t)}{dt} = \frac{b}{\alpha}x_1(t) - \frac{1}{M_0}x_2(t),$$

$$\frac{dx_2(t)}{dt} = \frac{1}{(1-\alpha)}x_2(t)\left(\frac{b}{\alpha} - \frac{(1-\alpha)}{\psi_2(t)\,x_2(t)} - \beta\left(1-\frac{p_0}{a}\frac{x_2^{(1-\alpha)}(t)}{x_1(t)}\right)\right),$$

$$\frac{d\psi_1(t)}{dt} = \rho\psi_1(t) - \frac{b}{\alpha}\psi_1(t) - \frac{p_0\,\beta}{a(1-\alpha)}\psi_2(t)\frac{x_2^{(2-\alpha)}(t)}{x_1^2(t)},$$

$$\frac{d\psi_2(t)}{dt} = \rho\psi_2(t) - \frac{b}{\alpha(1-\alpha)}\psi_2(t) + \frac{\beta}{(1-\alpha)}\psi_2(t) + \frac{1}{M_0}\psi_1(t) +$$

$$+\frac{(1-\alpha)}{x_2(t)}-\frac{(2-\alpha)}{(1-\alpha)}\frac{p_0\,\beta}{a}\frac{\psi_2(t)\,x_2^{(1-\alpha)}(t)}{x_1(t)}. \tag{43}$$

The Normalized Hamiltonian System

For providing economic interpretations of the Hamiltonian dynamics we introduce the following change variables

$$z_1=\psi_1\,x_1, \qquad z_2=\psi_2\,x_2 \tag{44}$$

for costs of material consumption x_1, x_2 by prices ψ_1, ψ_2, respectively.

For the Hamiltonian dynamics of costs z_1, z_2 in the case $(i=3)$ of the intermediate optimal control one can obtain the following system of differential equations

$$\frac{dx_1(t)}{dt}=\frac{b}{\alpha}x_1(t)-\frac{1}{M_0}x_2(t),$$

$$\frac{dx_2(t)}{dt}=\frac{1}{(1-\alpha)}x_2(t)\left(\frac{b}{\alpha}-\frac{(1-\alpha)}{z_2(t)}-\beta\left(1-\frac{p_0}{a}\frac{x_2^{(1-\alpha)}(t)}{x_1(t)}\right)\right),$$

$$\frac{dz_1(t)}{dt}=\rho\,z_1(t)-\frac{1}{M_0}z_1(t)\frac{x_2(t)}{x_1(t)}+\frac{p_0\,\beta}{a\,(1-\alpha)}z_2(t)\frac{x_2^{(1-\alpha)}(t)}{x_1(t)},$$

$$\frac{dz_2(t)}{dt}=\rho\,z_2(t)+\frac{1}{M_0}z_1(t)\frac{x_2(t)}{x_1(t)}-\frac{p_0\,\beta}{a}\,z_2(t)\frac{x_2^{(1-\alpha)}(t)}{x_1(t)}-\alpha. \tag{45}$$

Steady State

We are interested in existence of steady states for the Hamiltonian dynamics (45) which are connected with the structure of the optimal solution of the posed optimal control problem. The equilibrium conditions for steady states are presented by the system of algebraic equations

$$\frac{b}{\alpha}x_1-\frac{1}{M_0}x_2=0, \tag{46}$$

$$\frac{1}{(1-\alpha)}x_2\left(\frac{b}{\alpha}-\frac{(1-\alpha)}{z_2}-\beta\left(1-\frac{p_0}{a}\frac{x_2^{(1-\alpha)}}{x_1}\right)\right)=0, \tag{47}$$

$$\rho\,z_1-\frac{1}{M_0}z_1\frac{x_2}{x_1}+\frac{p_0\,\beta}{a\,(1-\alpha)}z_2\,\frac{x_2^{(1-\alpha)}}{x_1}=0, \tag{48}$$

$$\rho z_2 + \frac{1}{M_0} z_1 \frac{x_2}{x_1} - \frac{p_0 \beta}{a} z_2 \frac{x_2^{(1-\alpha)}}{x_1} - \alpha = 0 . \tag{49}$$

Let us note that the steady state solution can be considered as the "ideal" equilibrium state of the economic growth model at which the variables of material consumption x_1, x_2 and their costs z_1, z_2 keep constant equilibrium values.

We find the solution for the steady state equations (46) analytically under the following conditions – regularity conditions,

$$\beta > \rho > \frac{b}{\alpha} . \tag{50}$$

The first inequality in (50) means that the effectiveness coefficient β of investment in raising the resource productivity should be greater than the discount rate ρ . The second inequality in (50) presumes that the discount rate ρ is larger than the growth rate b of production factors since elasticity coefficient α is less than one, $\alpha < 1$.

The steady state $\left(x_1^*, x_2^*, z_1^*, z_2^*\right)$ as the solution of equilibrium equations (46)-(49) under conditions (50) has the following analytical form

$$x_1^* = \frac{\alpha}{b} \frac{1}{M_0} x_2^* = \frac{\alpha}{b} \frac{1}{M_0} \left(\frac{\beta \rho b \, p_0 M_0}{a(\alpha \rho - b)\left((\rho - b) + \alpha(\beta - \rho)\right)} \right)^{1/\alpha} , \tag{51}$$

$$x_2^* = \left(\frac{\beta \rho b \, p_0 M_0}{a(\alpha \rho - b)\left((\rho - b) + \alpha(\beta - \rho)\right)} \right)^{1/\alpha} , \tag{52}$$

$$z_1^* = -\left(1 + \frac{(\beta \alpha - b)}{\rho(1-\alpha)} \right) z_2^* = \frac{\alpha \left(1 + \frac{(\beta \alpha - b)}{\rho(1-\alpha)} \right)}{\left(\alpha(\beta - \rho) + \frac{b(\beta \alpha - b)}{\rho(1-\alpha)} \right)} , \tag{53}$$

$$z_2^* = -\frac{\alpha}{\left(\alpha(\beta - \rho) + \frac{b(\beta \alpha - b)}{\rho(1-\alpha)} \right)} . \tag{54}$$

Let us note that all coordinates of solution (51)-(54) have the property of well-posedness due to the regularity conditions (50).

It is important also to estimate the value of the optimal control u^* at the steady state $\left(x_1^*, x_2^*, z_1^*, z_2^*\right)$ (51)-(54)

$$u^* = \frac{b}{\alpha\,\beta}. \tag{55}$$

Due to regularity conditions (50) the value of the optimal control u^* is located in the proper range

$$0 < u^* < 1. \tag{56}$$

It means that it is reasonable to make an assumption that the upper bound u^0 for the control parameter u should satisfy to the following condition

$$\frac{b}{\alpha\,\beta} \le u^0 < 1. \tag{57}$$

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