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Optimization of Trends in Resource Productivity for Providing Sustainable Economic Development

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Abstract

In this presentation, a dynamic optimization model of investment in improvement of the resource productivity index is analyzed for obtaining balanced economic growth trends including both the consumption index and natural resources use.

The research is closely connected with the *problem of shortages of natural resources stocks*, the *security of supply of energy and materials*, and the *environmental effectiveness of their consumption*.

The main idea of the model is to introduce an integrated environment for elaboration of a control policy for management of the investment process in development of basic production factors such as capital, energy and material consumption.

An essential feature of the model is providing the possibility to invest in economy's dematerialization.

Important construction is connected with the price formation mechanism which presumes the rapid growth of prices on exhausting materials.

Abstract

The balance is formed in the consumption index which negatively depends on growing prices on materials.

The optimal control problem for the investment process is posed and solved within the Pontryagin maximum principle.

Specifically, the *growth* and *decline trends* of the *Hamiltonian trajectories* are examined for the optimal solution. It is proved that for specific range of the model parameters there exists the unique steady state of the Hamiltonian system.

The steady state can be interpreted as the optimal steady trajectory along which investments in improving resource productivity provide raising resource efficiency and balancing this trend with growth of the consumption index.

The comparison analysis is implemented for optimal model trends and historical trends of real econometric data.

As a *result of system analysis and modeling*, one can elaborate investment strategies in economy's dematerialization, resource and environmental management for improving the resource productivity index and, consequently, for shifting the economic system from non-optimal paths to the trajectory of sustainable development.

Introduction

The problem is to optimize trends in resources productivity and to balance investment in economy's dematerialization with sustainable growth of the consumption index.

The problem is considered within the classical approach *[Solow, 1970], [Schell, 1969]* of construction of economic growth models.

The main new element in the proposed model is a *price formation mechanism* which reflects possibility of rapid growth of prices on exhausting resources. Growing prices negatively influence on the *consumption index* which should be maximized in the model as *the basic element of the utility function*.

The stated problem has in its background very important concerns of the modern society with respect to the current world resource utilization. The recent statistics *[IPCC Report, 2007]*, *[OECD Report, 2008]* shows rapid increase of natural resource consumption, especially, in the following components: fossil energy (oil, natural gas, oil), ferrous metals (iron ore, etc.), nonferrous metals (bauxite, etc.), non-metalliferous minerals (lime), biomass (wood, etc.).

Taking into account the *limitations of natural resources*, at least, of its assured part, the *problem of raising resource efficiency and even reducing resource consumption becomes extremely significant*.

Introduction

Nowadays, a comprehensive research is being implemented on material flow analysis (MFA) by international (EUROSTAT, IPCC, OECD, World Resources Institute) and national (Germany, the Netherlands, the United States, Japan, China) research and policy making organizations.

Material flow analysis is a *systematic assessment of the flows and stocks of materials within a system defined in space and time*.

It connects
- the sources,
- the pathways,
- the intermediate and the final sinks of a material.

The method is an attractive decision-support tool in
- resource management,
- waste management,
- environmental management.

Introduction. References

Researches and develop the model of dynamic optimization of investment process in improving resource productivity are supplemented within the *economic growth theory*:

Arrow, K.J.
Production and Capital, 1985;

Ayres, R.U., Warr, B.
The Economic Growth Engine: How Energy and Work Drive Material Prosperity, 2009;

Barro, R.J., Sala-i-Martin, X.
Economic growth, 1995;

Crespo-Cuaresma, J., Palokangas, T., Tarasyev, A., (eds.)
Dynamic Systems, Economic Growth and the Environment, 2010.

Gordon, R.B., Koopmans, T.C., Nordhaus, W., Skinner, B.J.
Toward a New Iron Age? A Study of Patterns of Resource Exhaustion, 1988.

Grossman, G.M., Helpman, E.
Innovation and Growth in the Global Economy, 1991

Introduction. References

The construction of the model inherits elements of economic growth models introduced in papers:

Ane, B.K., Tarasyev, A.M., Watanabe, C.
Construction of Nonlinear Stabilizer for Trajectories of Economic Growth, 2007

Ane, B.K., Tarasyev, A.M., Watanabe, C.
Impact of Technology Assimilation on Investment Policy: Dynamic Optimization and Econometric Identification, 2007

Ayres, R., Krasovskii, A.A., Tarasyev, A.M.
Nonlinear Stabilizers of Economic Growth under Exhausting Energy Resources, 2009

Krasovskii, A.A., Tarasyev, A.M.
High-Precision Algorithms for Constructing Optimal Trajectories via Solving Hamiltonian Systems, 2009

Krasovskii, A.A., Kryazhimskiy, A.B., Tarasyev, A.M.
Optimal Control Design in Models of Economic Growth, 2008

Sanderson, W., Tarasyev, A.M., Usova, A.A.
Capital vs. Education: Assessment of Economic Growth from Two Perspectives, 2010

Tarasyev, A.M., Watanabe, C.
Optimal Dynamics of Innovation in Models of Economic Growth, 2001

Tarasyev, A.M., Watanabe, C., Zhu, B.
Optimal Feedbacks in Techno-Economic Dynamics,

Watanabe, C., Shin, J-H., Heikkinen, J., Tarasyev, A.
Optimal Dynamics of Functionality Development in Open Innovation, 2009

Introduction. References

Let us mention here papers are devoted to different aspects of economic growth modeling and conceptually are close to our approach:

Aseev, S., Besov, K., Kaniovski, S.

Optimal Endogenous Growth with Exhaustible Resources, 2010

Feichtinger, G., Hartl, R.F., Kort, P.M., Veliov, V.M.

Capital Accumulation under Technological Progress and Learning: a Vintage Capital Approach, 2006

Hutschenreiter, G., Kaniovski, Yu., Kryazhimskii, A.

Endogenous Growth, Absorptive Capacities and International R&D Spillovers, 1995

Introduction

The model dynamics includes the following main phase variables:

- production,
- current material use,
- cumulative material consumption.

Growing trend in production is given *exogenously* by the exponential term generated by such production factors as

- Capital,
- Labor.

Material use is introduced as a *production factor* in the *production function* of the Cobb-Douglas type.

The main control variable is presented by *investment in raising resource productivity* in the current period.

Prices on materials due to exhaustion are *growing rapidly to infinity* when the *cumulative material consumption is close to the available (assured) stock*.

In *the balance equation* both growth and decline trends are taken into account:

- *the growth trend* in the consumption index is stimulated by *the production growth*,
- *the decline trend* is caused by
  - *raising costs of materials* and
  - *expenditures directed on improvement of resource productivity*.

Introduction. Used approaches

The problem is to find the optimal proportion of investment in the dynamic process with maximization of the utility function given as the integrated consumption index over trajectories of the economic system.

The model is examined within the framework of the Pontryagin maximum principle

Pontryagin, L.S., Boltyanskii, V.G., Gamkrelidze, R.V., Mishchenko, E.F.

The Mathematical Theory of Optimal Processes, 1962

with special characteristics of infinite horizon

Aseev, S.M., Kryazhimskiy, A.V.

The Pontryagin Maximum Principle and Optimal Economic Growth Problems, 2007

Specific features of the corresponding Hamiltonian system are examined within the *qualitative theory of differential equations*

Hartman, Ph.

Ordinary Differential Equations, 1964

Introduction

In analysis we use *constructions of dynamic programming and the theory of generalized solutions of Hamilton-Jacobi equations*

Bellman, R.

Dynamic programming, 1957

Krasovskii, A.N., Krasovskii, N.N.

Control under Lack of Information, 1995

Subbotin, A.I.

Generalized Solutions for First-Order PDE. The Dynamical Optimization Perspective, Systems and Control: Foundations and Applications, 1995

Rockafellar, R.T.

Hamilton-Jacobi Theory and Parametric Analysis in Fully Convex Problems of Optimal Control, 2004

The range of model parameters is indicated for existence and uniqueness of a steady state.

The steady state plays the role of *the optimal steady solution* and *its proportions* can be used as an *economic standard* for the first approximation of solution of the optimal control problem.

At the steady state the optimal level of investment in resource productivity provides

- reduce in resource consumption,
- raise of its efficiency
- establish a reasonable balance between investment and consumption.

Introduction

We provide the comparison analysis and adjustment of optimal model trajectories to historical trends of real econometric data.

This analysis shows that

- the model quite adequately catches the main econometric tendencies and
- reflects the influence of investment in improvement of resource productivity on sustainable growth under limited resources.

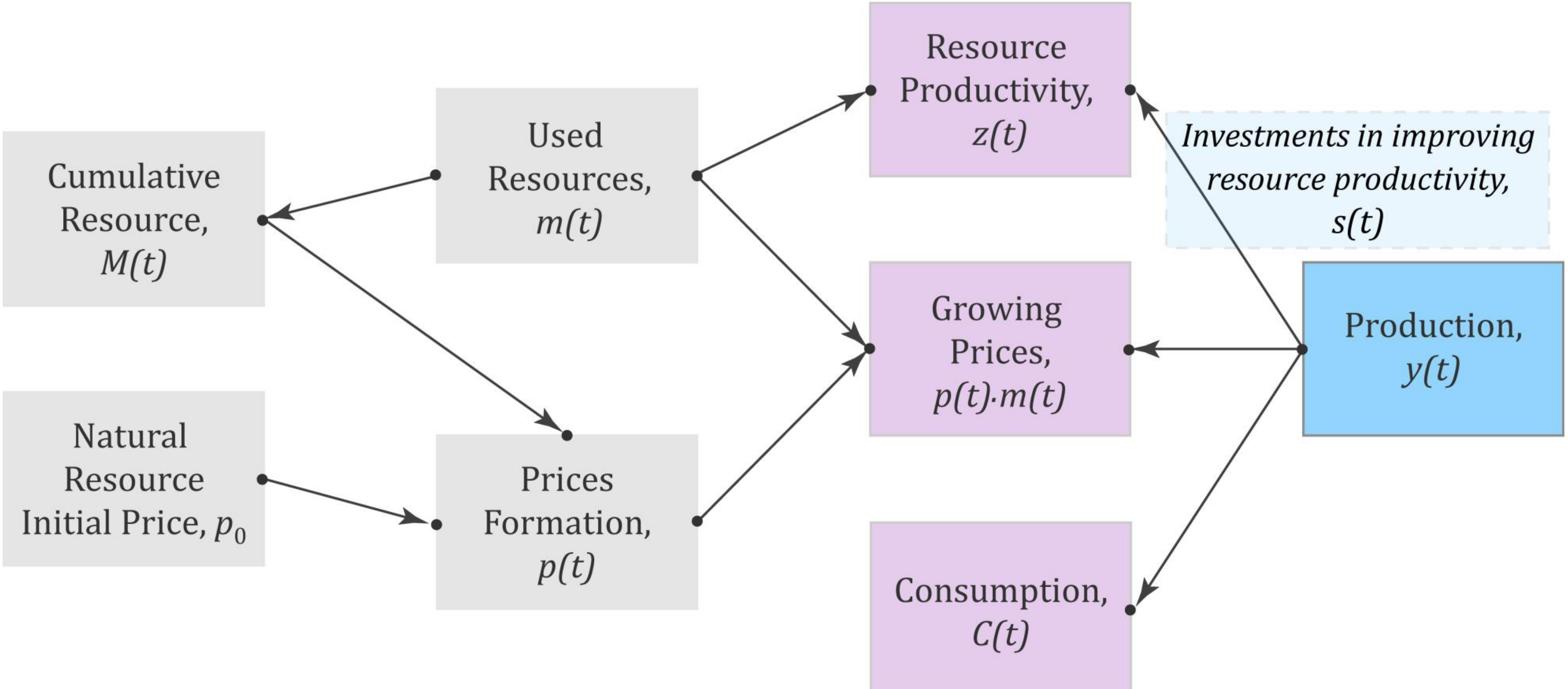
The main output of the implemented analysis and modeling is

- construction of investment strategies in economy's dematerialization and
- improvement of resource productivity.

The model simulations demonstrate that

> the proposed investment strategies could shift the economic system from non-optimal paths to the trajectory of sustainable development.

The Model. Scheme



The Model. Main variables

- Production: $y = y(t)$
- Used materials: $m = m(t), \;\; m(0) = m^*$
- Cumulative resource consumption:

$$M(t) = \int_0^t m(s)ds\,, \qquad M(0) = M^* = 0$$

- Resource productivity: $z(t) = \dfrac{y(t)}{m(t)}$
- Resource intensity: $Z(t) = \dfrac{1}{z(t)} = \dfrac{m(t)}{y(t)}$

- Rate of production: $\dfrac{dy(t)}{dt}$
- Rate of used materials: $\dfrac{dm(t)}{dt}$
- Rate of the cumulative resource consumption:

$$\frac{dM(t)}{dt} = m(t)$$

- Rate of resource productivity: $\dfrac{dz(t)}{dt}$

Price formation mechanism

Due to the limitation or exhaustion of natural materials its prices raise. It is assumed that prices are growing according to the inversely proportional rule of resources exhaustion:

$$p(t) = \frac{p_0}{\left(1 - \dfrac{M(t)}{M_0}\right)^{\gamma}},$$

M_0 – limitation of natural resources, p_0 – initial prices on natural resources,
γ – non-negative elasticity coefficient of the price formation mechanism

The Model. Balance equation

Production $y(t)$ in period t is shared between consumption $c(t)$ and the growing cost of natural resources $p(t) \cdot m(t)$ plus investment $s(t)$ in improving the resource productivity:

$$y(t) = c(t) + p(t) \cdot m(t) + s(t)$$

Investment level is bounded with positive constant parameter s^0:

$$0 \leq s(t) \leq s^0 < y(t)$$

Relative variables:

Consumption intensity:

$$\frac{c(t)}{y(t)}$$

Resource intensity:

$$Z(t) = \frac{m(t)}{y(t)}$$

Investment intensity:

$$u(t) = \frac{s(t)}{y(t)} \in [0, u^0]$$

Balance equation in relative variable

$$1 = \frac{c(t)}{y(t)} + p(t)Z(t) + u(t),$$

$$0 \leq u(t) \leq u^0 < 1$$

The Model. Production function and Consumption

An exponential production function of the Cobb-Douglas type:

$$y(t) = ae^{bt}\left(m(t)\right)^{\alpha}$$

Positive parameter a is a scale factor;

Non-negative growth rate b indicates the growth process of production $y(t)$ due to development of basic production factors such as capital, labor, technology, etc.;

The symbol α denotes non-negative elasticity coefficient of natural resources: $0 \leq \alpha < 1$.

A *production factor* is the diminishing return to scale of natural resources.

Consumption intensity

expressed through the resources consumption $m(t), M(t)$ by substituting relations of the price formation mechanism $p(t)$ and the production function $y(t)$ to the relative balance relation:

$$\frac{c(t)}{y(t)} = 1 - \frac{p_0}{ae^{bt}\left(1 - \frac{M(t)}{M_0}\right)^{\gamma}}\left(m(t)\right)^{1-\alpha} - u(t)$$

The Model. Dynamics

The relative raise in the resource productivity $z(t)$ is proportional to the portion of the assigned investment $u(t)$

$$\frac{1}{z(t)}\frac{dz(t)}{dt} = \beta u(t)$$

Non-negative parameter β describes the effectiveness of investments investment $u(t)$ in raising the resource productivity.

Since $z(t) = \frac{y(t)}{m(t)}$ the rate of the resource productivity can be decomposed into two components: the production rate $\frac{dy(t)}{dt}$ and the rate of the resource consumption $\frac{dm(t)}{dt}$:

$$\frac{1}{z(t)}\frac{dz(t)}{dt} = \frac{dy(t)}{y(t)} - \frac{dm(t)}{m(t)}$$

Due to the introduced production function last relation can be rewritten as follows:

$$\frac{1}{z(t)}\frac{dz(t)}{dt} = b - (1-\alpha)\frac{dm(t)}{m(t)} \Rightarrow \frac{dm(t)}{m(t)} = \frac{1}{1-\alpha}\left(b - \beta u(t)\right)$$

Hence, the rate of the resource consumption was defined.

The Model. Dynamics

The rate of the resource consumption

$$\frac{dm(t)}{m(t)} = \frac{1}{1-\alpha}\left(b - \beta u(t)\right)$$

is influenced by the production growth rate b and can be reduced only by investment $u(t)$ in raising the resource productivity.

> NOTE: If investment is equal to zero, $u(t) = 0$, then the rate of the resource consumption should be proportional to the production growth rate b.

New phase variables:

$$x_1(t) = e^{-\frac{b\gamma t}{1-\alpha-\gamma}}\left(1 - \frac{M(t)}{M_0}\right)^{\gamma}, \qquad x_2(t) = e^{-\frac{bt}{1-\alpha-\gamma}} m(t)$$

Dynamics of new variables:

$$\dot{x}_1(t) = -\frac{b\gamma}{1-\alpha-\gamma} x_1(t) - \frac{\gamma}{M_0}\left(x_1(t)\right)^{1-\frac{1}{\gamma}} x_2(t), \qquad x_1(0) = 1$$

$$\dot{x}_2(t) = \frac{1}{1-\alpha}\left(-\frac{b\gamma}{1-\alpha-\gamma} - \beta u(t)\right) x_2(t), \qquad x_2(0) = m^*$$

The Model. Logarithmic Consumption Index

Logarithmic consumption index in time t:

$$\ln c(t) = \ln a + \alpha \ln x_2(t) + \ln\left(1 - \frac{p_0}{a}\frac{(x_2(t))^{1-\alpha}}{x_1(t)} - u(t)\right) + \frac{(1-\gamma)b}{1-\alpha-\gamma}t$$

The integrated logarithmic index discounted with the discount rate $\rho, \ \rho > 0,$

$$J(x_1(\cdot), x_2(\cdot), u(\cdot)) = \int_0^T e^{-\rho t} \ln c(t)\, dt\,, \qquad 0 < T \leq +\infty$$

is the utility function in the considered control problem.

Optimal control problem

The problem is to maximize the utility function

$$J(x_1(\cdot), x_2(\cdot), u(\cdot)) = \int_0^T e^{-\rho t} \ln c(t)\, dt, \qquad 0 < T \le +\infty$$

over control processes $(x_1(t), x_2(t), u(t))$ of the dynamic system

$$\dot{x}_1(t) = -\frac{b\gamma}{1-\alpha-\gamma} x_1(t) - \frac{\gamma}{M_0} (x_1(t))^{1-\frac{1}{\gamma}} x_2(t), \qquad x_1(0) = 1$$

$$\dot{x}_2(t) = \frac{1}{1-\alpha}\left(-\frac{b\gamma}{1-\alpha-\gamma} - \beta u(t)\right) x_2(t), \qquad x_2(0) = m^*$$

satisfying denoted initial conditions and subject to constraints for the control parameter:

$$0 \le u(t) \le u^0 < 1.$$

NOTE: The main goal of the posed optimal control problem is to raise the resource productivity.

Optimal control problem. Special case

The problem is investigated for the special case when the elasticity coefficient γ in the price formation mechanism has the unit value, $\gamma = 1$.
Hence, phase variables x_1, x_2 are rewritten in the form:

$$x_1(t) = e^{\frac{b}{\alpha}t}\left(1 - \frac{M(t)}{M_0}\right), \qquad x_2(t) = e^{\frac{b}{\alpha}t} m(t)$$

and satisfy the dynamical system:

$$\dot{x}_1(t) = \frac{b}{\alpha} x_1(t) - \frac{1}{M_0} x_2(t), \qquad x_1(0) = 1$$

$$\dot{x}_2(t) = \frac{1}{1-\alpha}\left(\frac{b}{\alpha} - \beta u(t)\right) x_2(t), \qquad x_2(0) = m^*$$

Utility functional looks as follows:

$$J(x_1(\cdot), x_2(\cdot), u(\cdot)) = \int_0^T e^{-\rho t}\left(\alpha \ln x_2(t) + \ln\left(1 - \frac{p_0}{a}\frac{(x_2(t))^{1-\alpha}}{x_1(t)} - u(t)\right)\right) dt,$$

$$0 < T \leq +\infty.$$

Hamiltonian function

Hamiltonian function:

$$\widetilde{H}(x_1, x_2, u, t, \tilde{\psi}_1, \tilde{\psi}_2) = e^{-\rho t}\left(\alpha \ln x_2 + \ln\left(1 - \frac{p_0}{a}\frac{x_2^{1-\alpha}}{x_1} - u\right)\right) +$$
$$+ \tilde{\psi}_1\left(\frac{b}{\alpha}x_1 - \frac{1}{M_0}x_2\right) + \tilde{\psi}_2\frac{1}{1-\alpha}\left(\frac{b}{\alpha} - \beta u\right)x_2$$

where $\tilde{\psi}_1,\ \tilde{\psi}_2$ are adjoint variables.

The following change of conjugate variables and Hamiltonian function

$$\psi_1 = \tilde{\psi}_1 e^{\rho t}, \qquad \psi_2 = \tilde{\psi}_2 e^{\rho t}, \qquad H(x_1, x_2, u, \psi_1, \psi_2) = \widetilde{H}(x_1, x_2, u, t, \tilde{\psi}_1, \tilde{\psi}_2)e^{\rho t}$$

allows to obtain the expression for the stationary Hamiltonian:

$$H(x_1, x_2, u, t, \tilde{\psi}_1, \tilde{\psi}_2) = \alpha \ln x_2 + \ln\left(1 - \frac{p_0}{a}\frac{x_2^{1-\alpha}}{x_1} - u\right) +$$
$$+ \psi_1\left(\frac{b}{\alpha}x_1 - \frac{1}{M_0}x_2\right) + \psi_2\frac{1}{1-\alpha}\left(\frac{b}{\alpha} - \beta u\right)x_2.$$

Maximized Hamiltonian function

Maximization of the stationary Hamiltonian function with respect to the control variable u, $u \in [0, u^0]$ provides the following results:

$$H_1 = \alpha \ln x_2 + \ln\left(1 - \frac{p_0}{a}\frac{x_2^{1-\alpha}}{x_1}\right) + \psi_1\left(\frac{b}{\alpha}x_1 - \frac{1}{M_0}x_2\right) + \psi_2 \frac{1}{1-\alpha} x_2 \frac{b}{\alpha}, \qquad u = 0;$$

$$H_2 = \alpha \ln x_2 + \ln\left(1 - \frac{p_0}{a}\frac{x_2^{1-\alpha}}{x_1} - u^0\right) + \psi_1\left(\frac{b}{\alpha}x_1 - \frac{1}{M_0}x_2\right) + \psi_2 \frac{1}{1-\alpha} x_2 \left(\frac{b}{\alpha} - \beta\, u^0\right), \qquad u = u^0$$

$$H_3 = \alpha \ln x_2 + \ln\left(-\frac{1-\alpha}{\beta \psi_2 x_2}\right) + \psi_1\left(\frac{b}{\alpha}x_1 - \frac{1}{M_0}x_2\right) + \psi_2 \frac{1}{1-\alpha} x_2 \left(\frac{b}{\alpha} - \beta + \frac{\beta p_0}{a}\frac{x_2^{1-\alpha}}{x_1}\right) - 1, \quad u = u^*$$

Here parameter u^* is the intermediate maximum control value:

$$u^* = 1 - \frac{p_0}{a}\frac{x_2^{1-\alpha}}{x_1} + \frac{1-\alpha}{\beta \psi_2 x_2}.$$

For the intermediate control regime the adjoint variable ψ_2 is strictly negative.

Hamiltonian systems. Zero control regime

Due to the Pontryagin maximum principal one can obtain three Hamiltonian systems corresponding to different control regimes.

The Hamiltonian system for zero control regime $u = 0$:

$$\frac{dx_1(t)}{dt} = \frac{b}{\alpha} x_1(t) - \frac{1}{M_0} x_2(t),$$

$$\frac{dx_2(t)}{dt} = \frac{b}{\alpha(1-\alpha)} x_2(t),$$

$$\frac{d\psi_1(t)}{dt} = \left(\rho - \frac{b}{\alpha}\right)\psi_1(t) - \frac{p_0}{a} \frac{1}{1 - \frac{p_0}{a}\frac{x_2^{1-\alpha}(t)}{x_1(t)}} \frac{x_2^{1-\alpha}(t)}{x_1^2(t)},$$

$$\frac{d\psi_2(t)}{dt} = \frac{1}{M_0}\psi_1(t) + \left(\rho - \frac{b}{\alpha(1-\alpha)}\right)\psi_2(t) - \frac{p_0(1-\alpha)}{a} \frac{1}{1 - \frac{p_0}{a}\frac{x_2^{1-\alpha}(t)}{x_1(t)}} \frac{x_2^{-\alpha}(t)}{x_1(t)}$$

Hamiltonian systems. Upper bound control regime

The Hamiltonian system for the upper bound control regime $u = u^0 < 1$:

$$\frac{dx_1(t)}{dt} = \frac{b}{\alpha} x_1(t) - \frac{1}{M_0} x_2(t),$$

$$\frac{dx_2(t)}{dt} = \frac{1}{1-\alpha}\left(\frac{b}{\alpha} - \beta u^0\right) x_2(t),$$

$$\frac{d\psi_1(t)}{dt} = \left(\rho - \frac{b}{\alpha}\right)\psi_1(t) - \frac{p_0}{a} \frac{1}{1 - \frac{p_0}{a}\frac{x_2^{1-\alpha}(t)}{x_1(t)} - u^0} \frac{x_2^{1-\alpha}(t)}{x_1^2(t)},$$

$$\frac{d\psi_2(t)}{dt} = \frac{1}{M_0}\psi_1(t) + \left(\rho - \frac{b - \alpha\beta u^0}{\alpha(1-\alpha)}\right)\psi_2(t) - \frac{p_0(1-\alpha)}{a} \frac{1}{1 - \frac{p_0}{a}\frac{x_2^{1-\alpha}(t)}{x_1(t)} - u^0} \frac{x_2^{-\alpha}(t)}{x_1(t)} - \frac{\alpha}{x_2(t)}$$

Hamiltonian systems. Intermediate control regime

The Hamiltonian system for the intermediate optimal control regime

$$u = u^* = 1 - \frac{p_0}{a}\frac{x_2^{1-\alpha}}{x_1} + \frac{1-\alpha}{\beta\psi_2 x_2}:$$

$$\frac{dx_1(t)}{dt} = \frac{b}{\alpha}x_1(t) - \frac{1}{M_0}x_2(t),$$

$$\frac{dx_2(t)}{dt} = \frac{1}{1-\alpha}\left(\frac{b}{\alpha} - \frac{1-\alpha}{x_2(t)\psi_2(t)} - \beta\left(1 - \frac{p_0}{a}\frac{x_2^{1-\alpha}}{x_1}\right)\right)x_2(t),$$

$$\frac{d\psi_1(t)}{dt} = \left(\rho - \frac{b}{\alpha}\right)\psi_1(t) - \frac{p_0\beta}{a(1-\alpha)}\frac{x_2^{2-\alpha}(t)}{x_1^2(t)}\psi_2(t),$$

$$\frac{d\psi_2(t)}{dt} = \frac{1}{M_0}\psi_1(t) + \left(\rho - \frac{b-\alpha\beta}{\alpha(1-\alpha)} - \frac{(2-\alpha)}{(1-\alpha)}\frac{p_0\beta}{a}\frac{x_2^{1-\alpha}(t)}{x_1(t)}\right)\psi_2(t) + \frac{1-\alpha}{x_2(t)}$$

The Normalized Hamiltonian System

For providing economic interpretations of the Hamiltonian dynamics we introduce the following change variables: $z_1 = \psi_1 x_1$ and $z_2 = \psi_2 x_2$ for the costs of material consumption x_1, x_2 by prices ψ_1 and ψ_2, respectively.

The Hamiltonian system for the intermediate control regime $u = u^*$ in new adjoint variables has the form:

$$\frac{dx_1(t)}{dt} = \frac{b}{\alpha} x_1(t) - \frac{1}{M_0} x_2(t),$$

$$\frac{dx_2(t)}{dt} = \frac{1}{1-\alpha} x_2(t) \left(\frac{b}{\alpha} - \frac{1-\alpha}{z_2(t)} - \beta \left(1 - \frac{p_0}{\alpha} \frac{x_2^{1-\alpha}(t)}{x_1(t)} \right) \right),$$

$$\frac{dz_1(t)}{dt} = \left(\rho - \frac{1}{M_0} \frac{x_2(t)}{x_1(t)} \right) z_1(t) + \frac{p_0 \beta}{\alpha(1-\alpha)} \frac{x_2^{1-\alpha}(t)}{x_1(t)} z_2(t),$$

$$\frac{dz_2(t)}{dt} = \frac{1}{M_0} \frac{x_2(t)}{x_1(t)} z(t) + \left(\rho - \frac{p_0 \beta}{\alpha} \frac{x_2^{1-\alpha}(t)}{x_1(t)} \right) z_2(t) - \alpha$$

Steady State

Existence of the steady state is possible at the domain of the intermediate optimal control regime.

Steady state solution can be considered as the "ideal" equilibrium state of the economic growth model at which the variables of material consumption x_1, x_2 and their costs z_1, z_2 keep constant equilibrium values.

Steady state coordinates can be found analytically under the following assumption named as *regularity condition:*

$$\beta > \rho > \frac{b}{\alpha}$$

This condition means that

- the effectiveness coefficient β of investment is raising the resource productivity should be greater than the discount rate ρ,
- the discount rate ρ is larger than the growth rate b of production factors (since elasticity coefficient α is less than one, $\alpha < 1$).

Due to the regularity condition steady state coordinates have the *property of wellposedness*.

Optimal Control Value at the Steady State

Estimation of the optimal control value u^* at the steady state $(x_1^*, x_2^*, z_1^*, z_2^*)$:

$$u^* = \frac{b}{\alpha\beta}$$

Due to the *regularity condition* the value of the optimal control u^* is located in the proper range

$$0 < u^* < 1.$$

This fact means that it is reasonable to make an assumption:

> The upper bound u^0 for the control parameter u should satisfy to the following condition:
>
> $$\frac{b}{\alpha\beta} \leq u^0 < 1.$$

Optimization of Trends in Resource Productivity for Providing Sustainable Economic Development

Thank you for your attention!