Energy Balance Climate Models and General Equilibrium Optimal Mitigation Policies

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IIASA, Vienna 12/2011

Motivation

1. We study the economics of climate change by coupling spatial energy balance climate models (EBCM) in which heat diffuses across latitudes with economic growth models. This approach, integrates solution methods for spatial climate models, that may be new to economics, with methods of solving economic models, It provides new insights regrading the spatial distribution of temperature, damages relative to the more conventional integrated assessment models with carbon cycle which do not account heat transfer across latitudes.
2. Integrated assessment models (IAMs) include space in the sense of regional disaggregation (e.g.RICE 2010) but not heat transport across latitudes.
3. The interactions between heat diffusion and economic variables however could be important in characterizing the spatial distribution of damages due to climate change as well as the impacts of mitigation policies

Purpose-Energy Balance Climate Models (EBCM)

Characteristics

1. Explicit incorporation of the spatial dimension into the climate model in the form of heat diffusion or transport across latitudes.
    - one, two-dimensional models
2. The presence of an endogenous ice line where latitudes north (south) of the ice line are solid ice and latitudes south (north) of the ice line are ice free.
3. The underlying latitude dependent temperature function.

EBCMs

Potential output

The use of a spatial EBCMs model allows us to:

- Estimate a distribution of temperature anomaly across latitudes.
- Estimate the spatial effects of climate change by deriving a damage function that depends not on the average global temperature anomaly but on the distribution of temperature anomaly across latitudes.
- Introduce the concept "damage reservoirs" like ice-lines and permafrost as possible second component of damages in addition to the conventional components associated with conventional temperature increase. With damage reservoirs marginal damages are expected to be high initially and then decline as the ice-lines move to the Poles and permafrost disappears. Once the reservoir is exhausted there is no further damages

Purpose

1. Couple ECBMs models with economic growth models and develop general equilibrium one-dimensional unified models.
2. In the context of these models:
    1. Explore the impact of heat transport across latitudes regarding the endogenous spatial distribution temperature and damages, and derive latitude specific response functions and measures of spatial inequalities across latitudes.
    2. Provide insights regarding the optimal temporal profile for general equilibrium climate policies
3. Introduce the economics profession to the spatial EBCMs with heat transport as a potentially useful alternative to simple carbon cycle models.

A Basic Energy Balance Climate Model (North 1975)

$$\frac{\partial I(x,t)}{\partial t} = QS(x,t)\,\alpha(x,x_s(t)) - [I(x,t) - h(x,t)] + D\frac{\partial}{\partial x}\left[\left(1-x^2\right)\frac{\partial I(x.t)}{\partial x}\right]$$

$$I(x,t) = A + BT(x,t),\ A = 201.4W/m^2,\ B = 1.45W/m^2$$

where units of x are chosen so that $x = 0$ denotes the Equator, $x = 1$ denotes the North Pole, and $x = -1$ denotes the South Pole; Q is the solar constant divided by 2; $S(x)$ is the mean annual meridional distribution of solar radiation which is normalized so that its integral from -1 to 1 is unity; $\alpha(x, x_s(t))$ is the absorption coefficient which is one minus the albedo of the earth-atmosphere system, with $x_s(t)$ being the latitude of the ice line at time t; and D is a thermal diffusion coefficient that has been computed as $D = 0.649Wm^{-2\circ}C^{-1}$

Outgoing radiation is reduced by $h(x,t)$: accumulated carbon dioxide.

Heat Transport and Spatial Aspects of the EBCM

The heat flux which is modelled by the term: $D\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial I(x,t)}{\partial x}\right]$
The ice line is determined dynamically by the condition :

$$\begin{array}{ll} T > -10°\text{C} & \text{no ice line present at latitude } x \\ T < -10°\text{C} & \text{ice present at latitude } x \end{array}$$

and the ice line absorption drops discontinuously because the albedo jumps discontinuously. North (1975a) specifies the co-albedo function as:

$$\alpha(x, x_s) = \begin{cases} \alpha_0 = 0.38 & |x| > x_s \\ \alpha_1 = 0.68 & |x| < x_s \end{cases}$$

EBCM with Human Emissions

Human input: $h(x,t) = \sigma(x)\,\xi \ln \frac{M(t)}{M_0}$ where M_0 denotes the preindustrial and $M(t)$ the time t stock of carbon dioxide in the atmosphere, $\xi = 5.35$ (IPCC 2001).
The stock of the atmospheric carbon dioxide:

$$\dot{M}(t) = \int_{x=-1}^{x=1} \beta(x,t)\, q(x,t)\, dx - mM(t)\,,\ \ M(0) = M_0$$

Emissions are proportional to the amount of fossil fuels used.
The total stock of fossil fuel available is fixed or,

$$\int_{x=-1}^{x=1} q(x,t)\, dx = q(t)\ \ ,\ \ \int_0^{\infty} q(t) = R_0$$

where $q(t)$ is total fossil fuels used across all latitudes at time t, and R_0 is the total available amount of fossil fuels on the planet.

A Basic Energy Balance Climate Model: Summary I

$$\frac{\partial I(x,t)}{\partial t} = QS(x,t)\,\alpha(x,x_s(t)) - [I(x,t) - h(x,t)] + D\frac{\partial}{\partial x}\left[\left(1-x^2\right)\frac{\partial I(x.t)}{\partial x}\right]$$

$$I(x,t) = A + BT(x,t),\ A = 201.4W/m^2,\ B = 1.45W/m^2$$

$$h(x,t) = \sigma(x)\,\xi \ln\frac{M(t)}{M_0}$$

$$\dot{M}(t) = \int_{x=-1}^{x=1} \beta(x,t)\,q(x,t)\,dx - mM(t),\ M(0) = M_0$$

$$\int_{x=-1}^{x=1} q(x,t)\,dx = q(t)\ ,\ \int_0^\infty q(t) = R_0$$

Energy Balance Climate Model and Approximations

$$B\frac{\partial T(x,t)}{\partial t} = QS(x)\alpha(x,x_s) - [(A + BT(x,t)) - h(x,t)] + DB\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial T(x,t)}{\partial x}\right]$$

Approximation.
A satisfactory approximation of the solution for can be obtained by the so called two mode solution where $n = \{0, 2\}$.

$$\hat{T}(x,t) = \sum_{n \text{ even}} T_n(t) P_n(x)$$

The Two Mode Solution

$$\hat{T}(x,t) = T_0(t) + T_2(t)P_2(x)$$

$$B\frac{dT_0(t)}{dt} = -A - BT_0(t) + \int_{-1}^{1}\left[QS(x)\alpha(x,x_s) + \xi \ln \frac{M(t)}{M_0}\sigma(x)\right]dx$$

$$B\frac{dT_2(t)}{dt} = -B(1+6D)T_2(t) + \frac{5}{2}\int_{-1}^{1}\left[QS(x)\alpha(x,x_s) + \xi \ln \frac{M(t)}{M_0}\sigma(x)\right]P_2(x)dx$$

$$T_0(0) = T_{00},\, T_2(0) = T_{20},\, P_2(x) = \frac{(3x^2-1)}{2}$$

$$S(x) = 1 + S_2P_2(x)\;,\;\; S_2 = -0.482$$

The Impact of D and the Ice Line

If $D \longrightarrow \infty$, then the solution $T_2(t)$ of (**??**) vanishes.
Thus for a given diffusion $D < \infty$ the relative contribution of $T_2(t)$ to the solution $\hat{T}(t)$ can be regraded as an a measure of whether the heat transport is important in the solution of the problem.
The ice line solves

$$T_0(t) + T_2(t; D) P_2(x_s(t)) = T_s, \ \ T_s = -10°\text{C}$$

and is given by a solution of the ice line condition above, i.e.

$$x_s(t) = P_+^{-1}\left(\frac{T_s - T_0(t)}{T_2(t; D)}\right)$$

A Basic Energy Balance Climate Model: Summary II (Climate)

$$\hat{T}(x,t) = T_0(t) + T_2(t;D)P_2(x)$$

$$B\frac{dT_0(t)}{dt} = -A - BT_0(t) + \int_{-1}^{1}\left[QS(x)\alpha(x,\hat{T}(x,t)) + \xi \ln\frac{M(t)}{M_0}\sigma(x)\right]dx$$

$$B\frac{dT_2(t)}{dt} = -B(1+6D)T_2(t) + \frac{5}{2}\int_{-1}^{1}\left[QS(x)\alpha(x,\hat{T}(x,t)) + \xi \ln\frac{M(t)}{M_0}\sigma(x)\right]P_2(x)dx$$

$$T_0(t) + T_2(t;D)P_2(x_s(t)) = T_s, \; T_s = -10°\mathrm{C}$$

A Basic Energy Balance Climate Model: Summary II (Humans)

$$h(x,t) = \sigma(x)\,\xi \ln \frac{M(t)}{M_0}$$

$$\dot{M}(t) = \int_{x=-1}^{x=1} \beta(x,t)\, q(x,t)\, dx - mM(t), \;\; M(0) = M_0$$

$$\int_{x=-1}^{x=1} q(x,t)\, dx = q(t) \;\; , \;\; \int_0^{\infty} q(t) = R_0$$

Results from a Simplified Climate Model

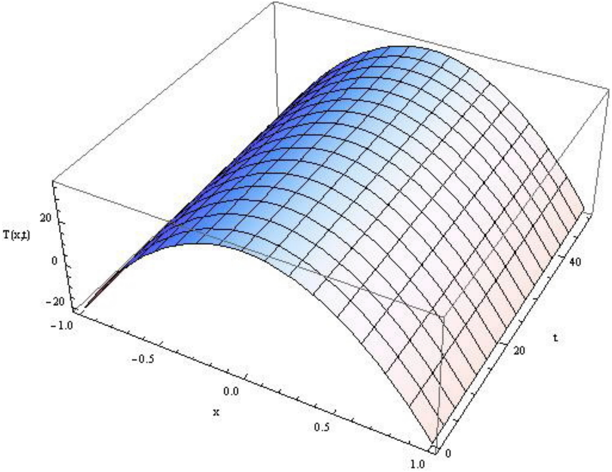
$a(x) = a_0 - a_1 P_2(x)$, that $S(x) = 1 - s_0 P_2(x)$ ($a_0 = 0.681$, $a_1 = 0.202$, $s_0 = 0.477$) (North at al. 1981).
The two-mode approximating ODEs become

$$\frac{dT_0}{dt} = -\frac{A}{B} - T_0(t) + \frac{1}{B}\left[\langle QS(x)\alpha(x), 1\rangle + \xi \ln \frac{M(t)}{M_0} \langle \sigma, 1\rangle\right]$$
$$\frac{dT_2}{dt} = -(1 + 6D)\, T_2(t) + \frac{5}{2B} \langle QS(x)\alpha(x), P_2(x)\rangle$$

Set $\frac{dT_0}{dt} = \frac{dT_2}{dt} = 0$. Then

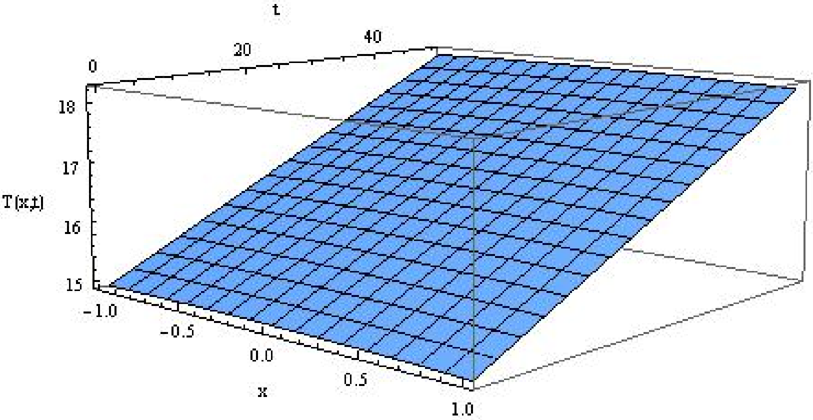
$$T(x, t; D) = C_0 + C_1 \ln \frac{M(t)}{M_0} - \frac{C_2}{(1 + 6D)} P_2(x) \quad , C_0, C_1, C_2 > 0$$

For an exogenous growth of the atmospheric CO_2 concentration, $M(t) = M_0 \exp(gt)$, we can obtain the spatial and temporal plot of the temperature function. $(g = 1.26\%)$



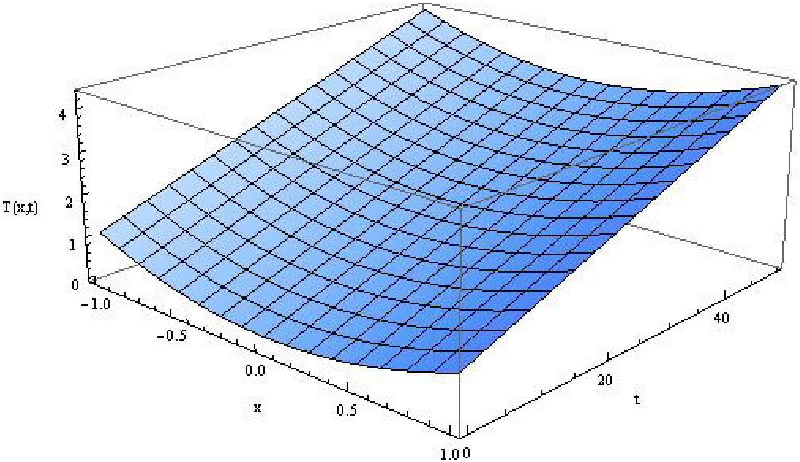
$T(0,0) = 36°C,\ T(1,0) = T(-1,0) = -23°C$

When $D \to \infty$ the temperature function is 'flat' across latitudes at the temperature of approximately 15°C



The Temperature Anomaly

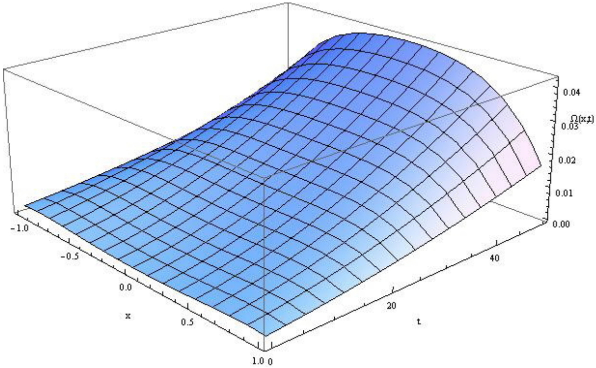
$T^{+}(x, t; D) = \hat{T}(x, t; D) - T_0(x, t)$, where $T_0(x, t)$ is distribution of temperature across latitudes as implied by existing data (NASA) for a benchmark period (1880-1900)



A Latitude Dependent Damage Function

$$\Omega\left(T^{+}(x,t;D)\right)=\frac{1}{1+\omega(x)\,\theta_1 T^{+}(x,t;D)+\omega(x)\,\theta_2\left(T^{+}(x,t;D)\right)^2}$$

$$1-\Omega\left(T^{+}(x,t;D)\right)$$



Climate Response Functions

$$
\begin{aligned}
dT_0(t) &= (\partial_{T_0,M})\, dM(t), \;\; dT_2(t) = (\partial_{T_2,M})\, dM(t) \\
dT(t,x) &= dT_0(t) + P_2(x)\, dT_2(t) = \\
& \left[(\partial_{T_0,M}) + (\partial_{T_2,M})\, P_2(x)\right] dM(t)
\end{aligned}
$$

The impact on damages will then be determined as:

$$
\begin{aligned}
d\Omega(T(x,t)) &= \Omega'_T \left[dT_0(t) + P_2(x)\, dT_2(t)\right] = \\
& \Omega'_T \left[(\partial_{T_0,M}) + (\partial_{T_2,M})\, P_2(x)\right] dM(t)
\end{aligned}
$$

An Economic EBC Model

$$
\begin{aligned}
Y(t,x) &= A(x,t)\Omega(T(x,t))F(K(x,t),L(x,t),q(x,t)) \\
&\equiv e^{(a+n\alpha_L)t}\Psi(x,T(x,t))K(x,t)^{\alpha_K}q(x,t)^{\alpha_q}
\end{aligned}
$$

The "*potential world GDP at date t*". (potential GDP).

$$
F_{total}(K(t),q(t),\{T(x,t)\}_{x=-1}^{x=1};x,t) = F_{total}(K(t),q(t),T;t)
$$

$$
C(t)+\dot{K}(t)+\delta K(t) = F_{total}(K(t),q(t),T;t),
$$

$$
j(t) = \int_{x=-1}^{x=1} j(x,t)\,dx, j = C,K,q
$$

$$
F_{total}(K(t),q(t),T;t) = \left[e^{(a+\alpha_L n)t}K(t)^{\alpha_K}q(t)^{\alpha_q}\right]J(t;D)
$$

$$
J\left(\{T(x,t)\}_{x=-1}^{x=1}\right) = J(t;D) \equiv \int_x \left\{\frac{\Psi(x,T(x,t))^{1/\alpha_L}}{\left[\int_{x'}\Psi(x',T(x',t)^{1/a_L}dx'\right]^{a_K+a_q}}\right\}
$$

Welfare Maximization in an EBCM

$$\max \int_0^\infty e^{-\rho t} \int_x L(x,t)[U\left(\frac{C(x,t)}{L(x,t)}\right) - \Omega_C(T(x,t))]dxdt$$

subject to:

- Climate dynamics
- Resource contrarian for the economy
- Total consumption and total fossil fuel constraints,
- States: $\mathbf{v} = (K(t), R(t), M(t), T(t,x))$,
- Controls: $\mathbf{u} = (C(t), C(x,t), q(t), q(x,t))$

Hamiltonian

$$
\begin{aligned}
\mathcal{H} \quad = \quad & \int_X L(x,t)[U\left(\frac{C(x,t)}{L(x,t)}\right) - \Omega_C(T(x,t))]dx + \\
& \lambda_K(t)\left[F_{total}(K(t), q(t), T; t) - C(t) - \delta K(t)\right] \\
& +\lambda_R(t)(-q(t)) + \lambda_M(t)\left[\int_{-1}^{1} \beta(t)\, q(x,t)\, dx - mM(t)\right] \\
& +\lambda_T(t,x)\left[\frac{1}{B}\left[QS(x)\alpha(x, T(x,t)) - (A + BT(x,t))\right.\right. \\
& -\sigma(x)\,\xi \ln \frac{M(t)}{M_0} + DB\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial T(x,t)}{\partial x}\right]\Bigg] \\
& +\mu_C(t)\left[C(t) - \int_X C(x,t)\,dx\right] + \mu_q(t)\left[q(t) - \int_X q(x,t)dx\right]\Bigg]
\end{aligned}
$$

Maximum Principle-Controls

$$
\begin{aligned}
C(t), C(x,t) &: \quad \lambda_K(t) = \mu_C(t) = U'\left(\frac{C(x,t)}{L(x,t)}\right) \\
q(t) &: \quad \lambda_K(t) F'_{total,q} = \lambda_R(t) - \mu_q(t) \\
q(x,t) &: \quad \lambda_M(t)\beta(t) = \mu_q(t) \\
\text{or } F'_{total,q} &= \frac{\lambda_R(t) - \lambda_M(t)\beta(t)}{\lambda_K(t)},
\end{aligned}
$$

Maximum Principle - Costates

$$\dot{\lambda}_K(t) = \left[\rho + \delta - F'_{total,K}(K(t), q(t), T; t)\right] \lambda_K(t)$$
$$\dot{\lambda}_R(t) = \rho \lambda_R(t)$$
$$\dot{\lambda}_M(t) = (\rho + m) \lambda_M(t) + \frac{\xi}{BM(t)} \int_{-1}^{1} \sigma(x) \lambda_T(t, x)$$
$$\dot{\lambda}_T(t, x) = (\rho + 1) \lambda_T(t, x) + L(t, x) \Omega'_{c,T}(T(t, x))$$
$$-\lambda_K(t) F'_{total,T}(K(t), q(t), T; t) -$$
$$QS(x) \frac{\lambda_T(t, x)}{B} \frac{\partial \alpha(x, T(x, t))}{\partial T} - D \frac{\partial}{\partial x} \left[(1 - x^2) \frac{\partial \lambda_T(x, t)}{\partial x} \right]$$

Optimal Paths

$$\left\{K^*\left(t;D\right), K^*\left(t,x;D\right) R^*\left(t;D\right), M^*\left(t;D\right), T^*\left(t,x;D\right)\right\}_{x=-1}^{x=1}$$

$$\left\{C^*\left(t;D\right), C^*\left(x,t;D\right), q^*\left(t;D\right), q^*(x,t;D)\right\}_{x=-1}^{x=1}$$

$$\left\{\lambda_K^*\left(t;D\right), \lambda_R^*\left(t;D\right), \lambda_M^*\left(t;D\right), \lambda_T^*\left(t,x;D\right)\right\}_{x=-1}^{x=1}$$

$$\max_{\{C(x,t)\}} \{ \int_{t=0}^{\infty} e^{-\rho t} L(x,t) U\left(\frac{C(x,t)}{L(x,t)}\right) - \Omega_C(T(x,t;D)) dt \}$$

subject to

$$C(x,t) + \dot{K}(x,t) + \dot{B}(x,t) = r(t)(K(x,t) + B(x,t)) + I(x,t)$$
$$B(x,0) = 0, K(x,0) = K_0(x)$$
$$I(x,t) \equiv w(x,t)L(x,t) + s_{FF}(x,t)\pi_{FF}(t) + s_{Tax}(x,t)Tax(t)$$

FONC

$$U'\left(\frac{C(x,t))}{L(x,t}\right) = \Lambda(x) \exp(\rho t - \int_{s=0}^{t} r(s) ds)$$

$\Lambda(x) = \Lambda(x')$ endowment reshuffling

Equilibrium

Consumption goods producing firms

$$\max \; \{A(x,t)\Omega(T(x,t))F(K(x,t),L(x,t),q(x,t)) - \\ (r(t)+\delta)K(x,t) - w(x,t)L(x,t) - p(t)q(x,t)\}$$

Optimality conditions:

$$\begin{aligned} F'_K(K(x,t),L(x,t),q(x,t),T(x,t)) &= r(t)+\delta \\ F'_q(K(x,t),L(x,t),q(x,t),T(x,t)) &= p(t) \end{aligned}$$

In market equilibrium

$$\dot{\lambda}_K(x,t) = \left(\rho + \delta - F'_K\right)\lambda_K(x,t)$$

Equilibrium

Fossil fuel firms

$$\max_{q(t)} \int_{t=0}^{\infty} \exp(-\int_{s=0}^{t} r(s)ds)[(p(t)-\theta(t))q(t)(1-\tau(t))])dt,$$

$$\text{subject to} \quad \int_{t=0}^{\infty} q(t)dt \leq R_0$$

FONC

$$\begin{aligned}(p(t)-\theta(t))(1-\tau(t)) &= \mu_0 \exp(\int_{s=0}^{t} r(s)ds), \text{ or} \\ (F'_q-\theta(t))(1-\tau(t)) &= \mu_0 \exp(\int_{s=0}^{t} r(s)ds)\end{aligned}$$

Market Equilibrium

Consumption goods firms at latitude x will choose demands $K(x,t)$ and $q(x,t)$ to set

$$r(t) + \delta = F'_K, \;\; p(t) = F'_q$$

For a multiplier value $\bar{\mu}_0$ that exhaust the fossil fuels reserves and parametric temperature

$$\{K(x,t;T), K(t;T), q(x,t;T), q(t;T)\}_{x=-1}^{x=1}$$

Substituting the paths into carbon and temperature dynamics will provide the corresponding paths for carbon dioxide accumulation and temperature.

$$\{K^e(x,t;D), K^e(t;D), q^e(x,t;D), q^e(t;D), R^e(t;D)\}_{x=-1}^{x=1}$$
$$\{M^e(t;D), T^e(t,x;D)\}_{x=-1}^{x=1}$$

Optimal Taxation

Intertemporal endowment flows $I(x,t)$s have been augmented so that

$$\lambda_K(x,t) = \lambda_K(x',t) \equiv \lambda_K^*(t;D)$$

Optimal taxes can be determined by equating private and social marginal products for fossil fuel

$$\begin{aligned} \theta^*(t) &= \frac{\lambda_R^*(t;D) - \lambda_M^*(t;D)\beta}{\lambda_K^*(t;D)} - \frac{\mu_0 e^{\Gamma(t)}}{(1-\tau^*(t))} \\ \exp(\int_{s=0}^t r(s)ds) &= e^{\Gamma(t)} \end{aligned}$$

The Temporal Profile of Optimal Taxes

Lemma: $\zeta(t) \equiv \int_x \sigma\lambda_T^*(t,x;D)dx < 0$, $\lambda_M^*(t;D) < 0$.

Proposition: If $m < \delta$, then the optimal profit tax decreases through time, or $\dot{\tau}^*(t) < 0$. Furthermore, the optimal unit tax on fossil fuels grows at a rate less than the rate of interest, or $\frac{\dot{\theta}^*(t)}{\theta^*(t)} < r^*(t)$.

A decreasing $\tau(t)$ implies

$$r^*(t) - \frac{\dot{p}^*(t)}{p^*(t)} > 0,$$

$$r^* - \frac{\dot{p}^*}{p^*} = \rho - \frac{\dot{\lambda}_K^*}{\lambda_K^*} - \frac{d\left[\left(\lambda_R^* - b\lambda_M^*\right)/\lambda_K^*\right]/dt}{\left[\left(\lambda_R^* - b\lambda_M^*\right)/\lambda_K^*\right]},$$

or using the optimality conditions for the costate variables

$$r^* - \frac{\dot{p}^*}{p^*} = \frac{\left[\delta\lambda_R^* + \beta\lambda_M^*(m-\delta) - \left(\beta\xi/BM(t)\right)\int_X \sigma\lambda_T^* dx\right]}{\left(\lambda_R^* - b\lambda_M^*\right)}$$

The impact of co-albedo and thermal transportation

The discounting function effect

$\int_x \sigma\lambda_T(t,x)\,dx \equiv \zeta$ the global shadow cost of temperature across latitudes: It holds $\dot{\zeta} = \upsilon\zeta - \Xi(t)$, where

$$\upsilon \equiv \rho + 1 - \frac{Q}{B}\left(\frac{\int_x \sigma\lambda_T(x,t;D)\,S(x)\,(\partial a/\partial T)\,dx}{\int_x \sigma\lambda_T(x,t;D)\,dx}\right)$$

Since $\partial a/\partial T > 0$ the discounting function υ falls. Forward discounted costs of climate change higher than when for the co-albedo function $\partial a/\partial T = 0$.

The damage effect

$$J(t;\infty) = \Omega(T_0(t;\infty)) \int_x \left\{ \frac{\int_x [A(x,0)^{1/\alpha_L} L(x,0)]}{\int_{x'} A(x',0)^{1/\alpha_L} L(x',0)dx']^{\alpha_K+\alpha_q}} \right\} dx$$

Impact of $D := |J(t;\infty) - J(t;D)|$

Approximations and Numerical Simulations

We simplify the economic model by assuming: (i) linear utility, which means $\gamma \longrightarrow 1$, and (ii) that the stock of capital relaxes fast, relative to the rest of the system to its steady state. These assumptions allows us to write the problem as a most rapid approach path (MRAP) problem. The objective becomes

$$\int_{t=0}^{\infty} e^{-\rho t} C(t) dt - \left[\int_{t=0}^{\infty} e^{(-\rho+n)t} \int_{x} L(x,0) \Omega_C(T(x,t) dx \right] dt,$$

For a two-mode approximation with fast relaxing modes we can write $T(t,x;D) = \psi(M(t);x,D)$. Define the damage functions as:

$$\begin{aligned} \int_x L(x,0)\Omega_C(T(x,t;D)dx &= D_C(M(t);D), \\ \int_x \left\{ \frac{\Psi(x,T(x,t);D)^{1/\alpha_L}}{\left[\int_{x'} \Psi(x',T(x',t;D)^{1/a_L} dx'\right]^{a_K+a_q}} \right\} dx &= D_F(M(t);D) \end{aligned}$$

The Social Optimum

$$\max_{K,q} \int_{t=0}^{\infty} e^{-\rho t} \left[e^{(a+na_L)t} K(t)^{a_K} q(t)^{a_q} D_F(M(t); D) \right.$$

$$\left. - (\rho + \delta) K(t) \right] dt - \left[\int_{t=0}^{\infty} e^{(-\rho+n)t} D_C(M(t); D)\, dt \right]$$

subject to

$$\dot{M} = \beta q - mM \;,\;\; M(0) = M_0$$

$$\int_{0}^{\infty} q(t) \leq \; R_0$$

Socially optimal time paths: $\left\{ K^S(t), q^S(t), M^S(t), \lambda_M^S(t) \right\}$ and the multiplier Λ_q

Socially Optimal Paths

$$K^S(t) = c_K \exp\left(\frac{a + na_L}{a_L} t\right) \left(\Lambda_q - \beta\lambda_M\right)^{-\frac{a_q}{a_L}} D_F\left(M(t); D\right)^{\frac{1}{a_L}}$$

$$c_K \equiv \left\{\left[\left(\frac{a_q}{a_K}\right)(\rho + \delta)\right]^{a_q} \frac{a_K}{\rho + \delta}\right\}^{1/a_L}, c_q \equiv \left(\frac{a_q}{a_K}\right)(\rho + \delta)\, c_K$$

$$q^S(t) = c_q \exp\left[\left(\frac{a + na_L}{a_L}\right) t\right] \left(\Lambda_q - \beta\lambda_M\right)^{-1-\frac{a_q}{a_L}} D_F\left(M(t); D\right)^{\frac{1}{a_L}}$$

$$\dot{M}(t) = -mM(t) + \beta q^S(t), \;\; M(0) = M_0$$

$$\dot{\lambda}_M = (\rho + m)\lambda_M + e^{nt} D'_{C,M}\left(M(t); D\right) -$$

$$e^{(a+na_L)t} K^S(t)^{a_K} q^S(t)^{a_q} D'_{F,M}\left(M(t); D\right)$$

$$\Lambda_q \left[\int_0^\infty q^S(t)\, dt - R_0\right] = 0, \;\; \Lambda_q \geq 0$$

The Laissez Faire Problem

Equilibrium time paths for $\{K(t;D), q(t;D), M(t;D)\}$ can be determined as:

$$K^e(t;D) = p(0)^{-\frac{a_q}{a_L}} c_K D_F(M(t);D)^{\frac{1}{a_L}} \exp\left(\frac{a + na_L - \rho a_q}{a_L} t\right)$$

$$c_K \equiv \left\{\left[\left(\frac{a_q}{a_K}\right)(\rho+\delta)\right]^{a_q} \frac{a_K}{\rho+\delta}\right\}^{1/a_L}$$

$$q^e(t;D) =$$

$$p(0)^{-1-\frac{a_q}{a_L}} c_q D_F(M(t);D)^{\frac{1}{a_L}} \exp\left[\left(\frac{a + na_L - \rho a_q}{a_L} - \rho\right) t\right]$$

$$c_q \equiv \left(\frac{a_q}{a_K}\right)(\rho+\delta) c_K$$

$$\dot{M}(t) = -mM(t) + \beta q^e(t), \ \ M(0) = M_0$$

$$\int_{s=0}^{\infty} q^e(s)\, ds \leq R_0$$

Inequalities Across Latitudes

Latitude specific damages:

$$L(x,0)\Omega_C\left(T^e(x,t;D)\right) = D_C^x\left(x,t;D\right),$$

$$\frac{\Psi\left(x,T^e(x,t;D)\right)^{1/\alpha_L}}{\left[\int_{x'}\Psi\left(x',T^e(x',t;D\right)^{1/a_L}dx'\right]^{a_K+a_q}} = D_F^x\left(x,t;D\right).$$

Latitude Specific Welfare and Production

$$W\left(x;D\right) = \int_{t=0}^{\infty} e^{-\rho t}\left[e^{(a+na_L)t}K^e\left(t,x;D\right)^{a_K} q^e\left(t,x;D\right)^{a_q} D_F^x\left(M^e\left(t;D\right)\right) -\delta K^e\left(t,x;D\right)\right]dt - \left[\int_{t=0}^{\infty} e^{(-\rho+n)t} D_C^x\left(M^e\left(t;D\right)\right)dt\right]$$

$$Q\left(x;D\right) = \int_{t=0}^{\infty} e^{-\rho t}\left[e^{(a+na_L)t}K^e\left(t,x;D\right)^{a_K} q^e\left(t,x;D\right)^{a_q} D_F^x\left(M^e\left(t;D\right)\right)\right]dt$$

Inequalities Across Latitudes

Latitude Specific Social Cost of Carbon - Output Cost of Carbon

$$\frac{\partial W\left(x;D\right)}{\partial M^{e}\left(t;D\right)},\frac{\partial Q\left(x;D\right)}{\partial M^{e}\left(t;D\right)}$$

Consumption - Production Related Damages

$$\begin{aligned} D_{C}\left(x;D\right) &= \int_{t=0}^{\infty} e^{(-\rho+n)t} D_{C}^{x}\left(M^{e}\left(t;D\right)\right) dt \\ D_{F}\left(x;D\right) &= \int_{t=0}^{\infty} e^{-\rho t} D_{F}^{x}\left(M^{e}\left(t:d\right)\right) dt \end{aligned}$$

Numerical Simulations - Laissez Faire Equilibrium

- Use of 2667 Giga Tons of Carbon in 150 years
- Current temperature of 36°C for the equator and -19°C for the Poles.
- Temperature without taxes increases uniformly by approximately 8 degrees within 150 years