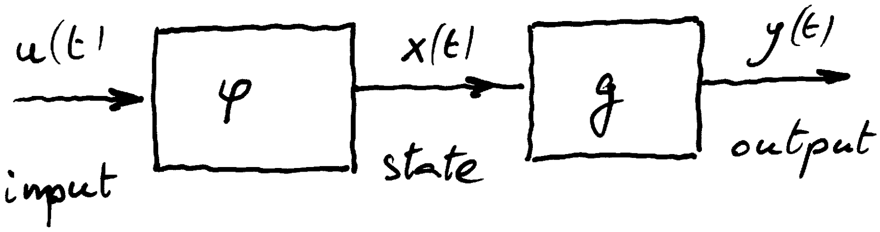
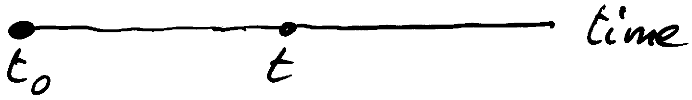
DYNAMICAL SYSTEMS





$$\left.\begin{array}{l} x(t) = \varphi\big(t_0, t, \underset{\substack{\uparrow \\ x(t_0)}}{x_0}, u_{[t_0,t)}(\cdot)\big) \\ \\ y(t) = g(x(t)) \end{array}\right\} \text{abstract definition}$$

x_0 is the information about the past necessary to predict the future

t = real $\Updownarrow$ continuous time

t = integer $\Updownarrow$ discrete time

Operational definition

$$u(t) \longrightarrow \boxed{\;f\;} \xrightarrow{x(t)} \boxed{\;g\;} \xrightarrow{y(t)}$$

discrete time

$$x(t) = \begin{vmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{vmatrix} \qquad x_i \in \mathbb{R} \qquad \Longrightarrow$$

$$x(t+1) = f\big(x(t), u(t)\big)$$

Example (Fibonacci)

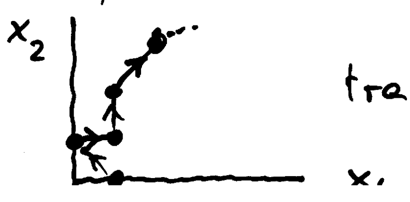
$$x_1(t+1) = x_2(t)$$

$$x_2(t+1) = x_1(t) + x_2(t) + u(t)$$

For $u(t) \equiv 0 \qquad x_0 = \begin{vmatrix} 1 \\ 0 \end{vmatrix}$

$$y(t) = x_1(t) + x_2(t)$$

1, 1, 2, 3, 5, 8, 13, ...



trajectory

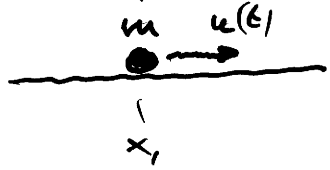
continuous time

...

...

$$\dot{x}(t) = f\big(x(t), u(t)\big)$$

Example (Newton)

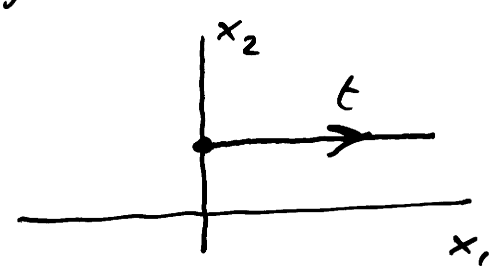


$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = \frac{1}{m}\, u(t)$$

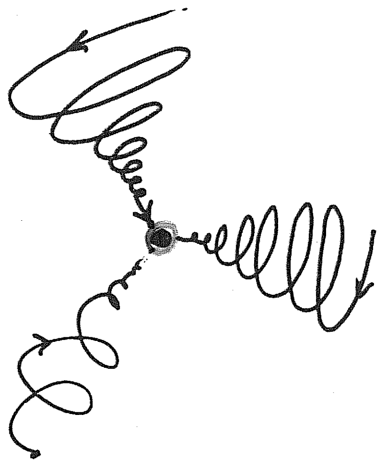
For $u \equiv 0 \qquad x_0 = \begin{vmatrix} 0 \\ 1 \end{vmatrix}$

$$y(t) = x_1(t) = t$$

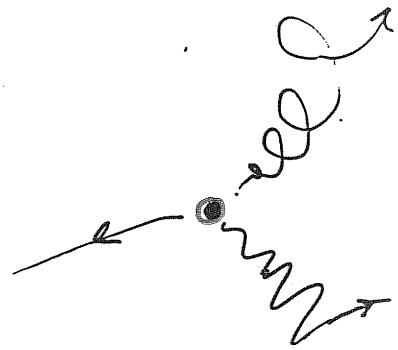


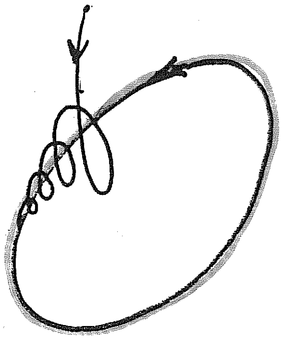
trajectory

ASYMPTOTIC BEHAVIOUR

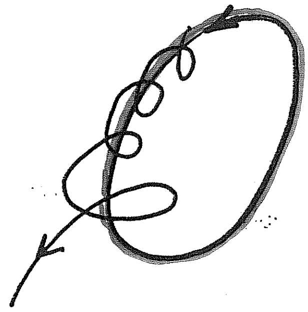


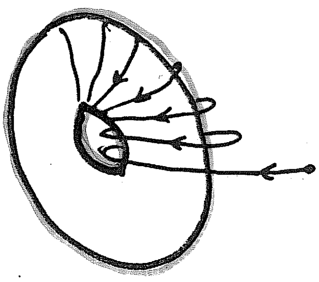
EQUILIBRIUM



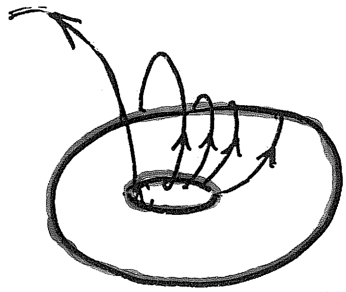


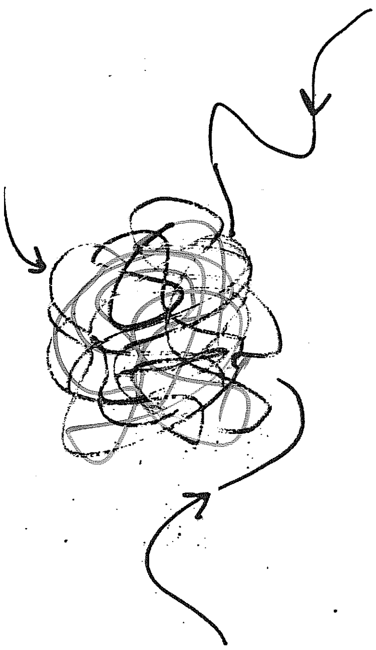
CYCLE





TORUS

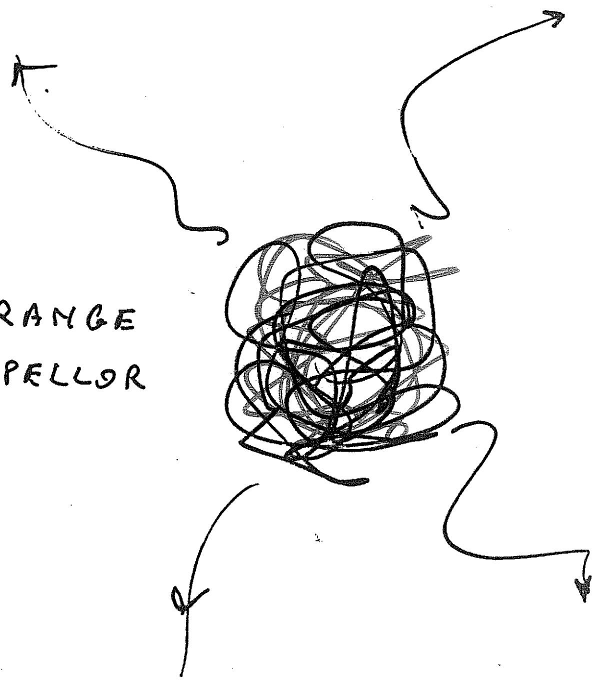




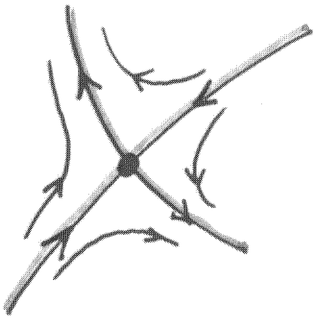
STRANGE
ATTRACTOR

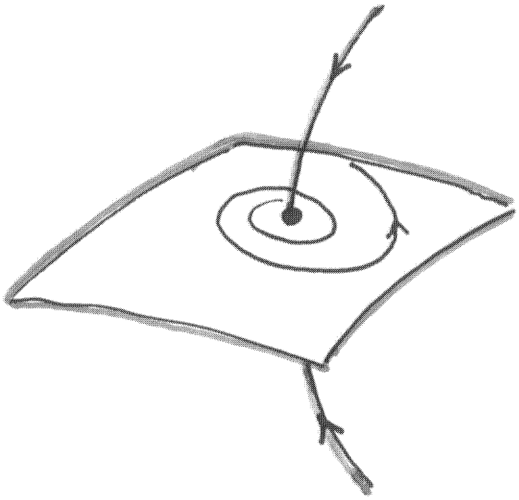
Lorenz system

STRANGE
REPELLOR

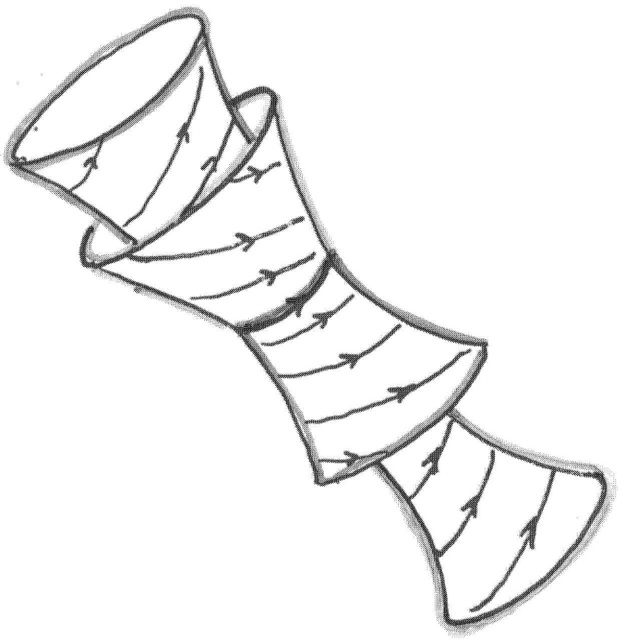


SADDLES





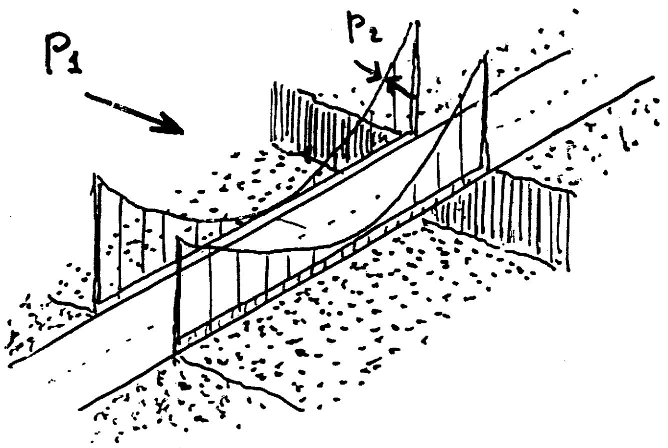
saddle equilibrium

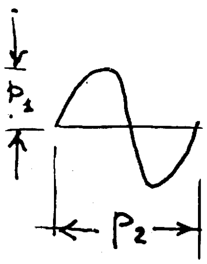


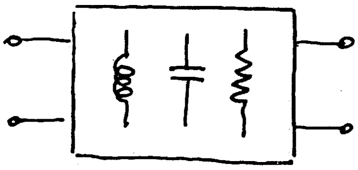
saddle cycle

strange saddle

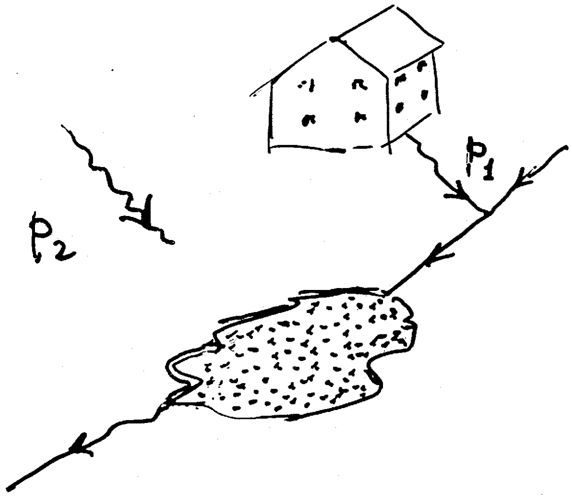
FAMILIES OF DYNAMICAL SYSTEMS

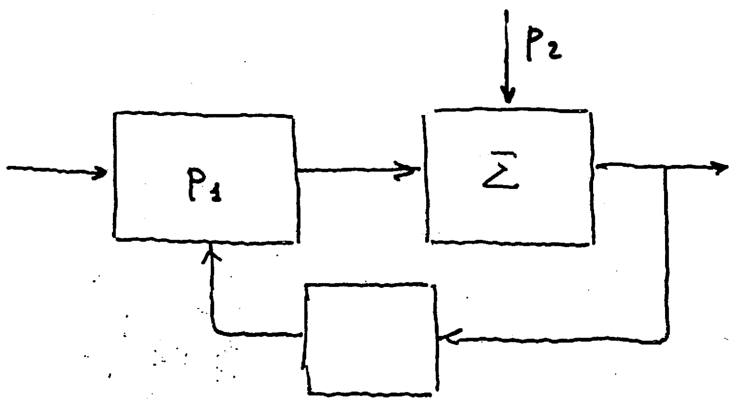






$$\begin{cases} \dot{x}_1(t) = f_1\left(x_1(t) \dots x_n(t),\ p_1 \dots p_k\right) \\ \vdots \\ \dot{x}_n(t) = f_n\left(x_1(t) \dots x_n(t),\ p_1 \dots p_k\right) \end{cases}$$

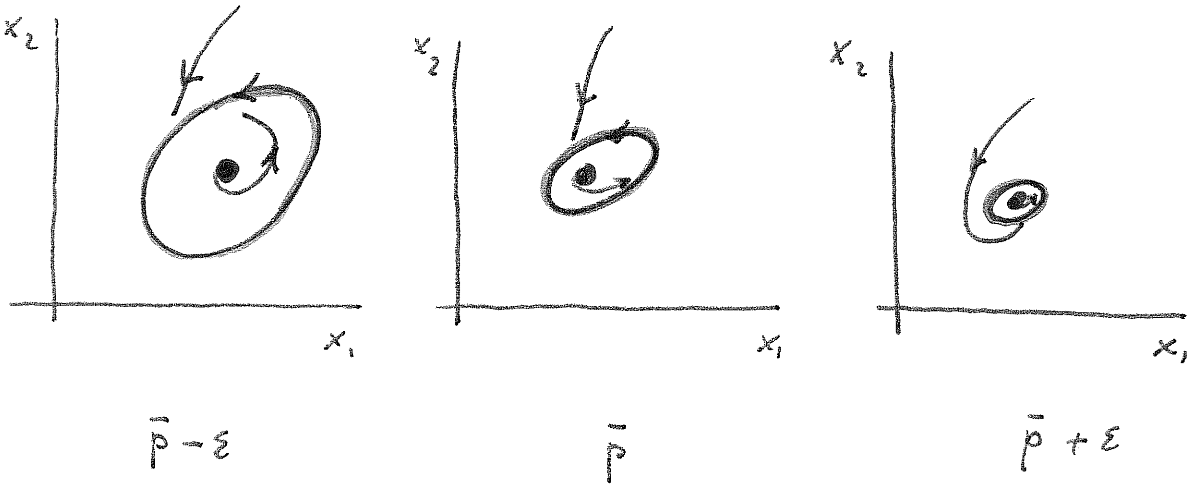




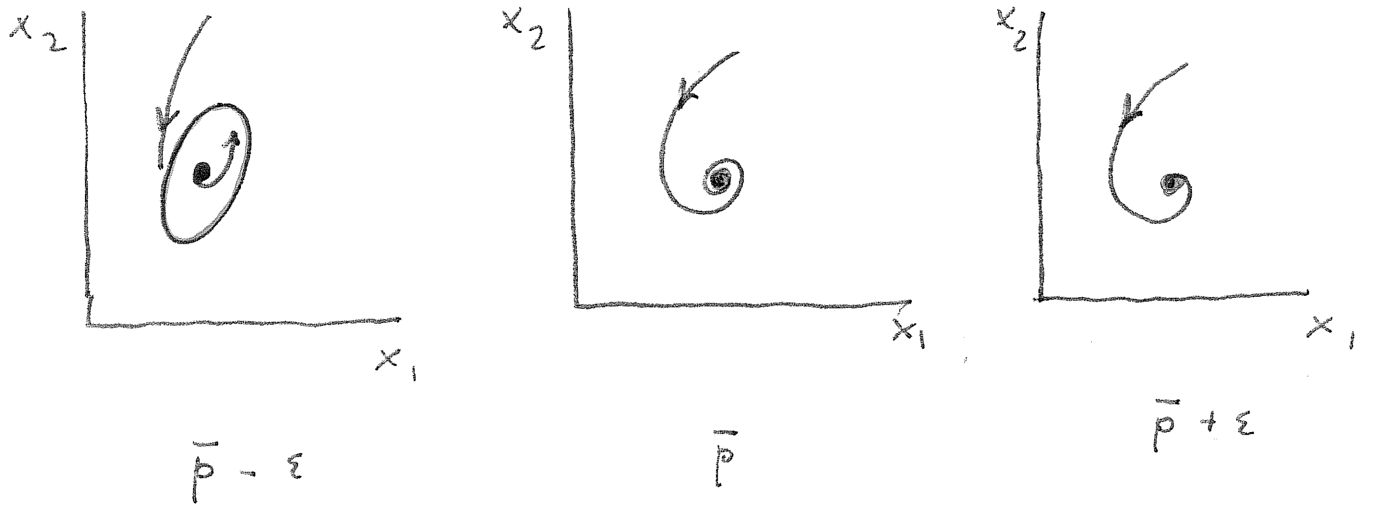
STRUCTURAL STABILITY

$\dot{x} = f(x, p)$

↑ constant parameter

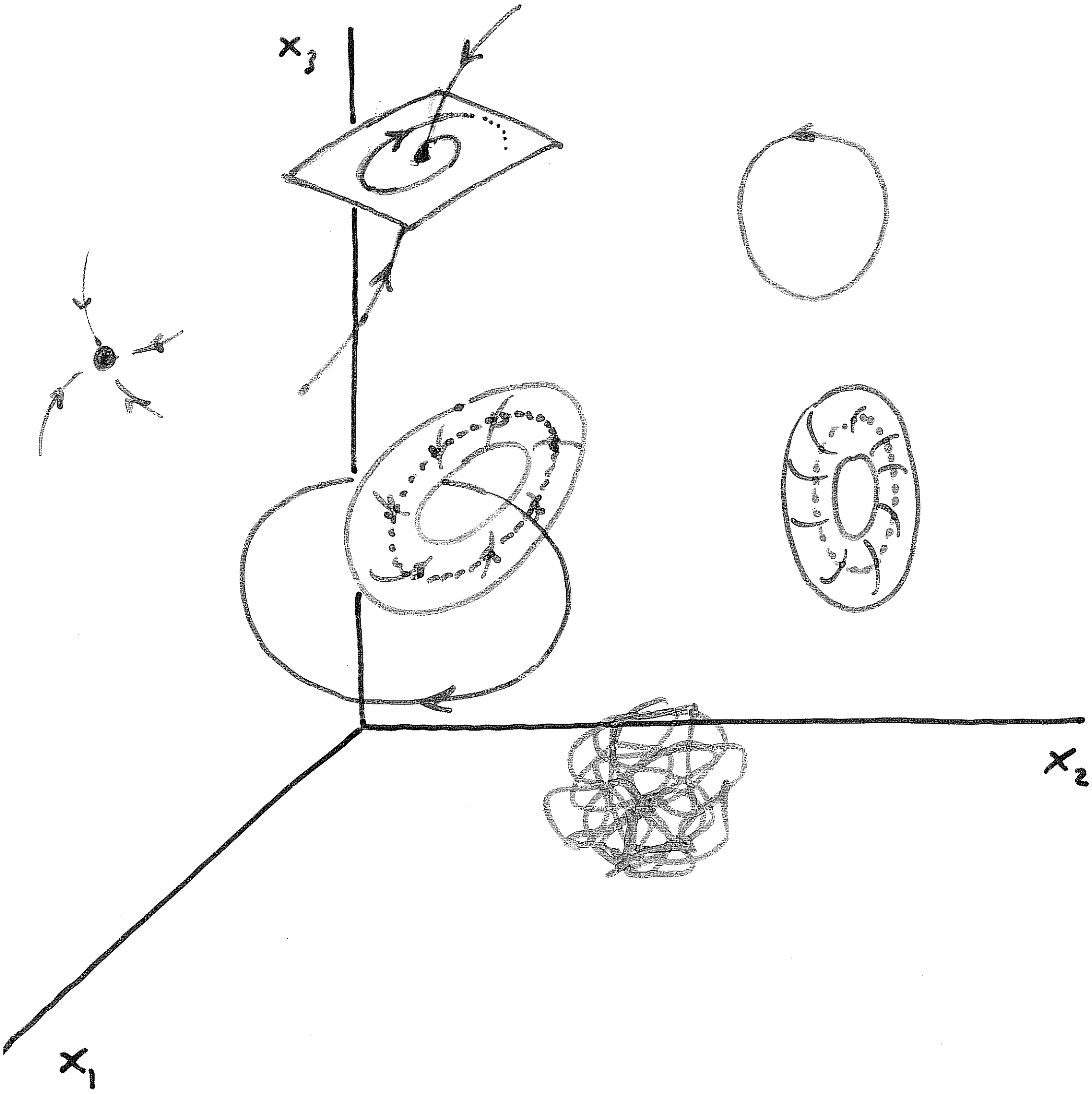


The system is structurally stable at $\bar{p}$ (state portraits are topologically equivalent)



The system is <u>not</u> structurally stable at $\bar{p}$ (state portraits are <u>not</u> topologically equivalent)

Bifurcations as collisions

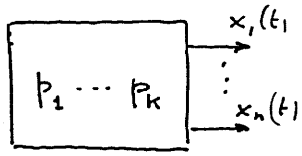


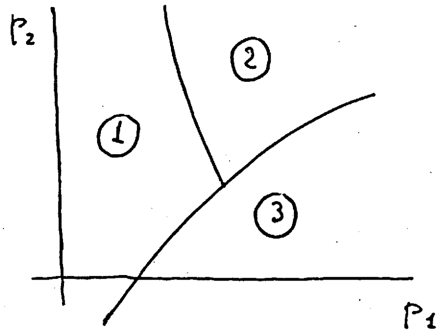
The system is structurally stable if attractors, repellors and saddles (and their stable and unstable manifolds) are "separated".

In fact, in such a case, a small variation of the parameters implies a small variation of the invariant sets, which remain separated, so that the portrait of the system remains qualitatively the same

<u>Conclusion</u> bifurcation $\approx$ collision of invariant sets

BIFURCATION DIAGRAMS





bifurcation diagram

In each region (i) the system has the same asymptotic behaviour.

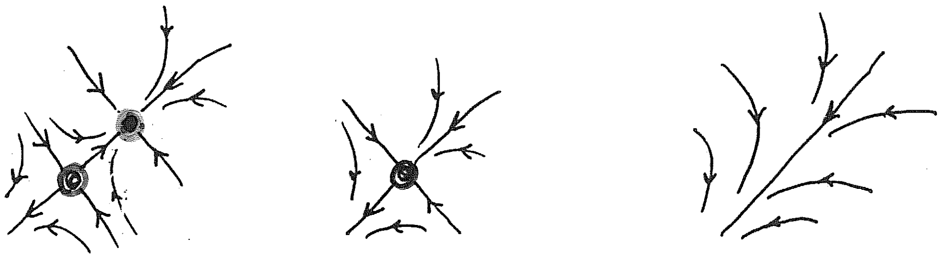
Points on the boundaries of the regions are points at which the system is <u>not structurally stable</u> : these points are called <u>bifurcation points</u>

TARGET   Bifurcation diagram and state portrait of each region (i)

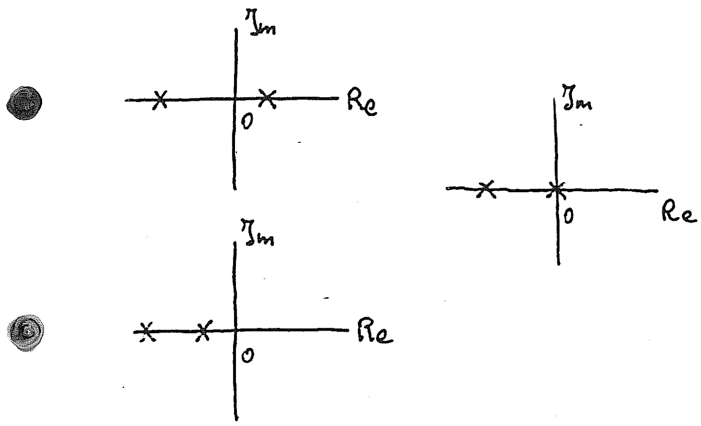
<u>COMPLEXITY</u> : number of regions (i)

BIFURCATIONS

$$\begin{cases} \text{LOCAL} \\ \text{GLOBAL} \end{cases} \qquad \begin{cases} \text{CATASTROPHIC} \\ \text{NON CATASTROPHIC} \end{cases}$$



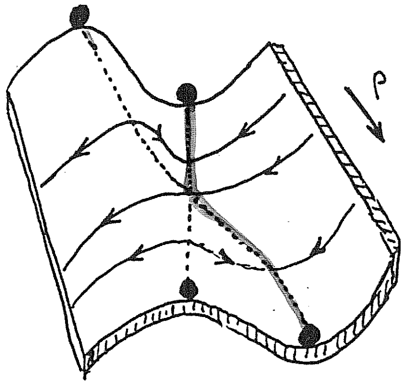
$p = \bar{p} - \varepsilon$ $\qquad$ $p = \bar{p}$ $\qquad$ $p = \bar{p} + \varepsilon$

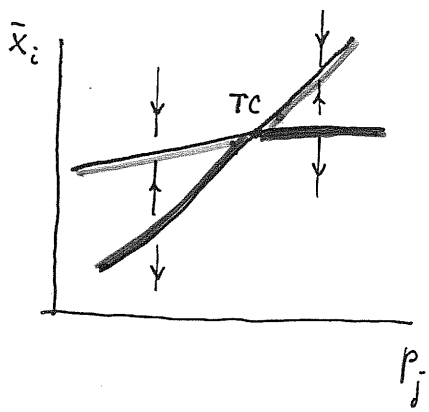


$$\text{Local bifurcations} \begin{cases} \text{collision of equilibria, cycles, ...} \\ \text{eigenvalues on stability boundary} \end{cases}$$

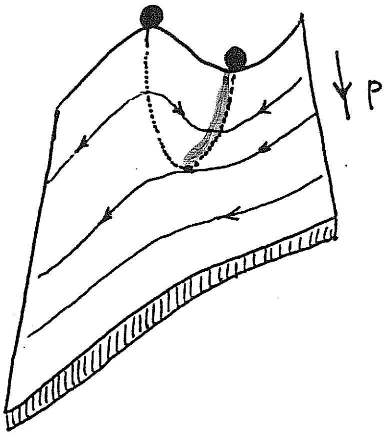
BIFURCATIONS OF EQUILIBRIA

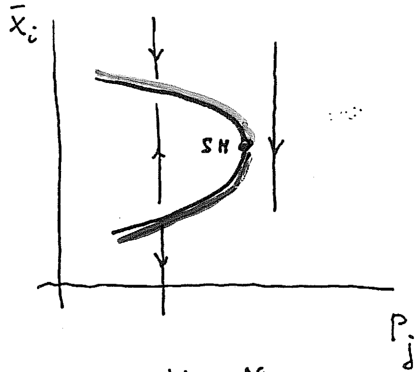
Transcritical (exchange of stability)





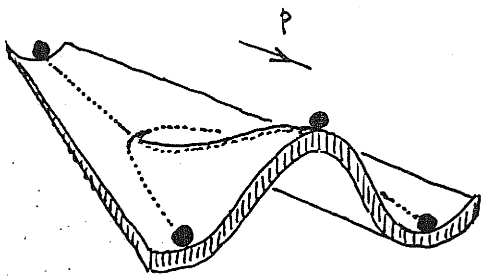
Saddle-Node (fold) (tangent)

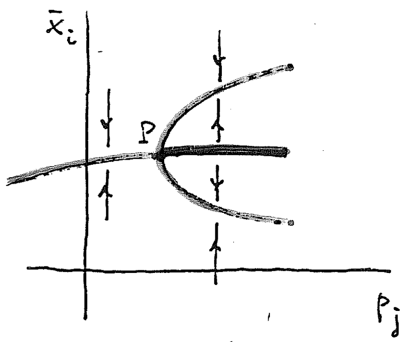




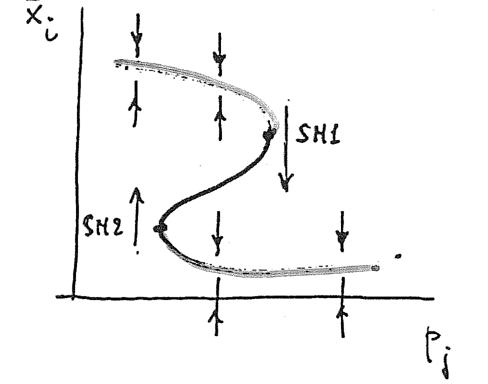
the flea

Pitchfork

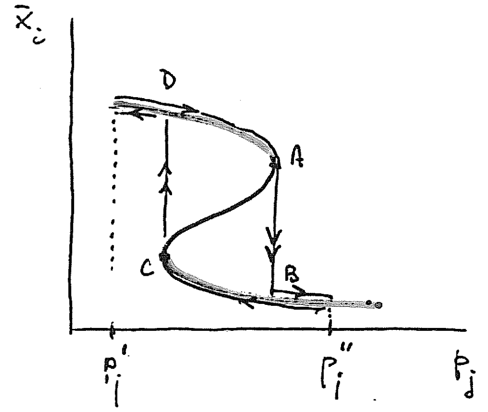




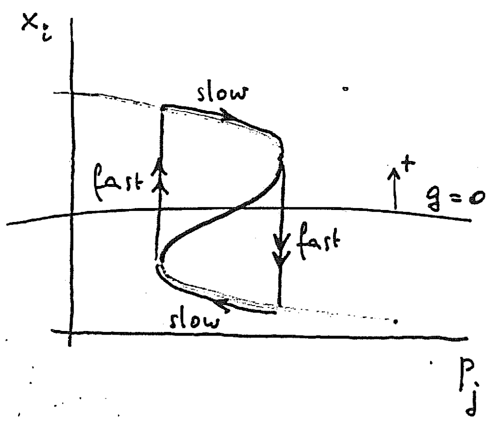
HYSTERESIS



Examples: resources - consumers
drug dealers - policemen
turbidity - flow rate
⋮
perceiptive catastrophes
holes or stones ?
the top
young or old ?
Zeeman machine



If p_j is varied slowly up and down we obtain catastrophic transitions AB and CD



If p_j varies autonomously

$$\dot{p}_j = \varepsilon g(p_j\ x_i) \qquad \varepsilon > 0 \text{ small}$$

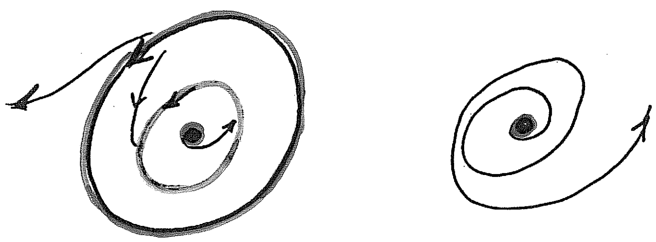
and $g = 0$ separates the stable branches of the hysteresis, we have a <u>slow-fast cycle</u>

BIFURCATIONS OF CYCLES

Hopf $n \geq 2$

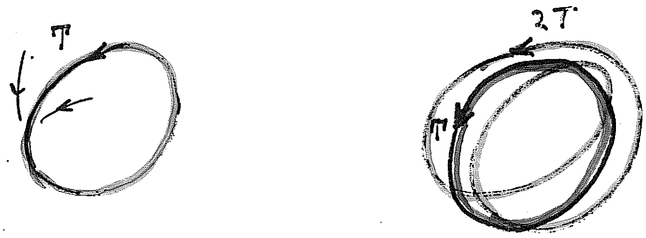
Parkinson's disease

Tangent of cycles $n \geq 2$



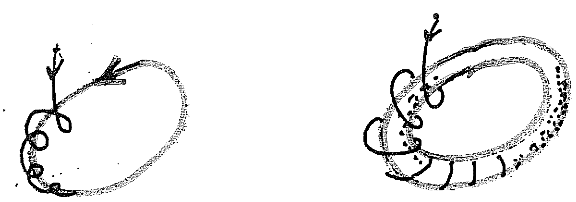
Tacoma's bridge

Flip (period doubling) $n \geq 3$



off-shore structures

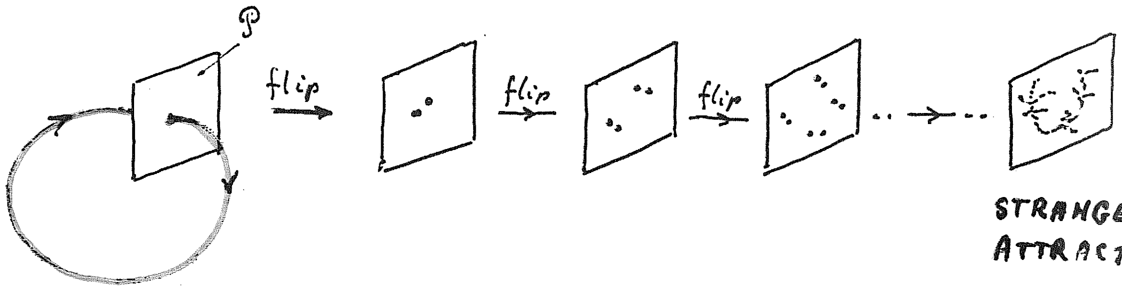
Naimark - Sacker $n \geq 3$

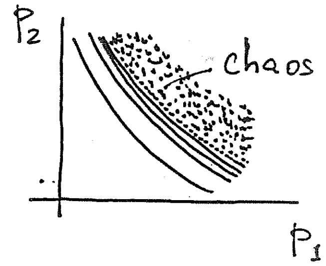


forced oscillators
sail boat

ROUTES TO CHAOS

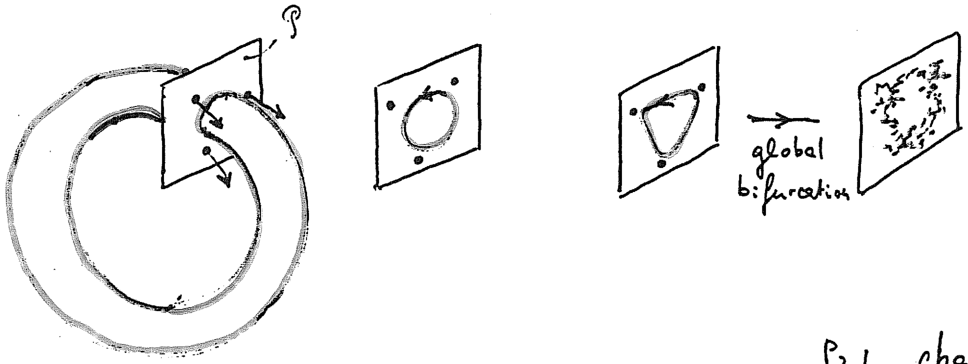
1. Feigenbaum's cascade of period doublings

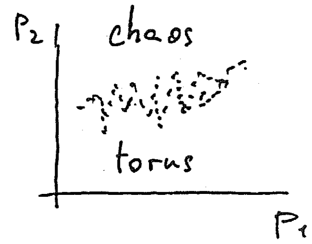




- ∞ local bifurcations
- universal accumulation law

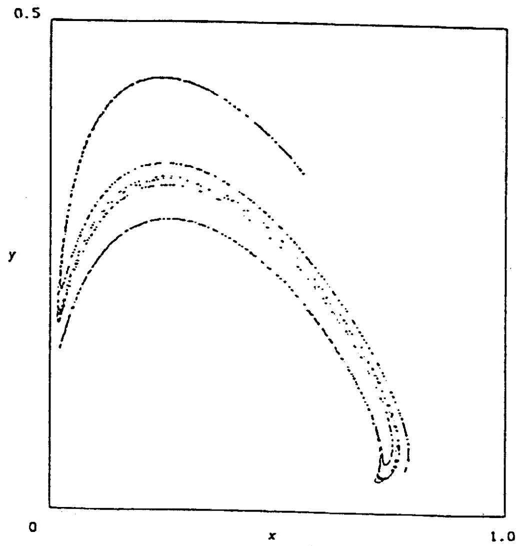
2. Torus destruction



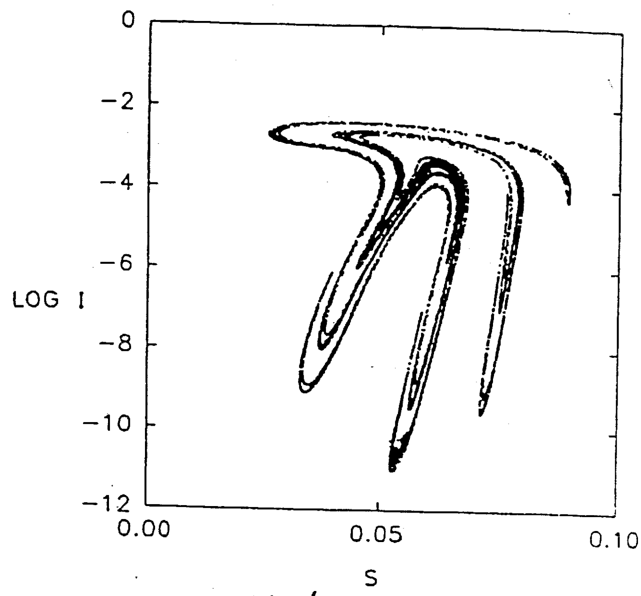


- only one global bifurcation
- fractal bifurcation set

EXAMPLES

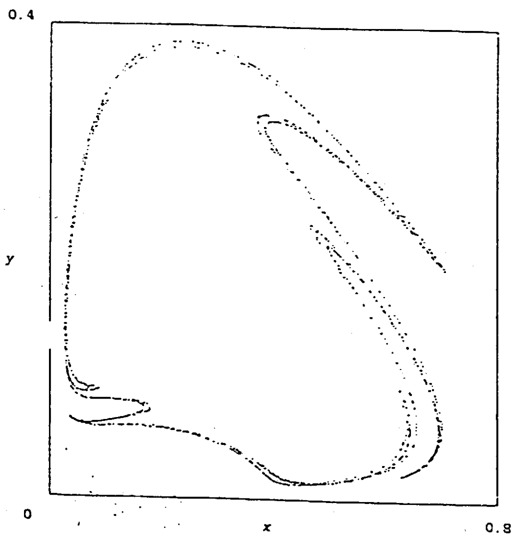


prey - predator

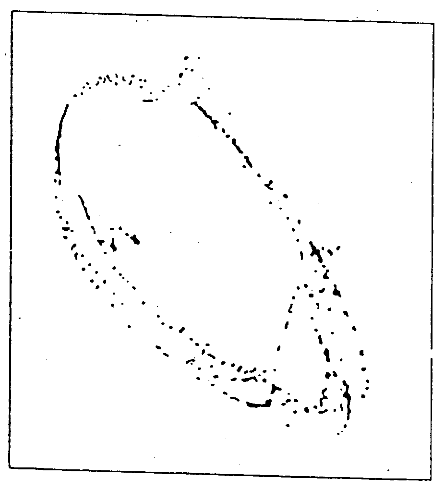


epidemics

fractal geometry

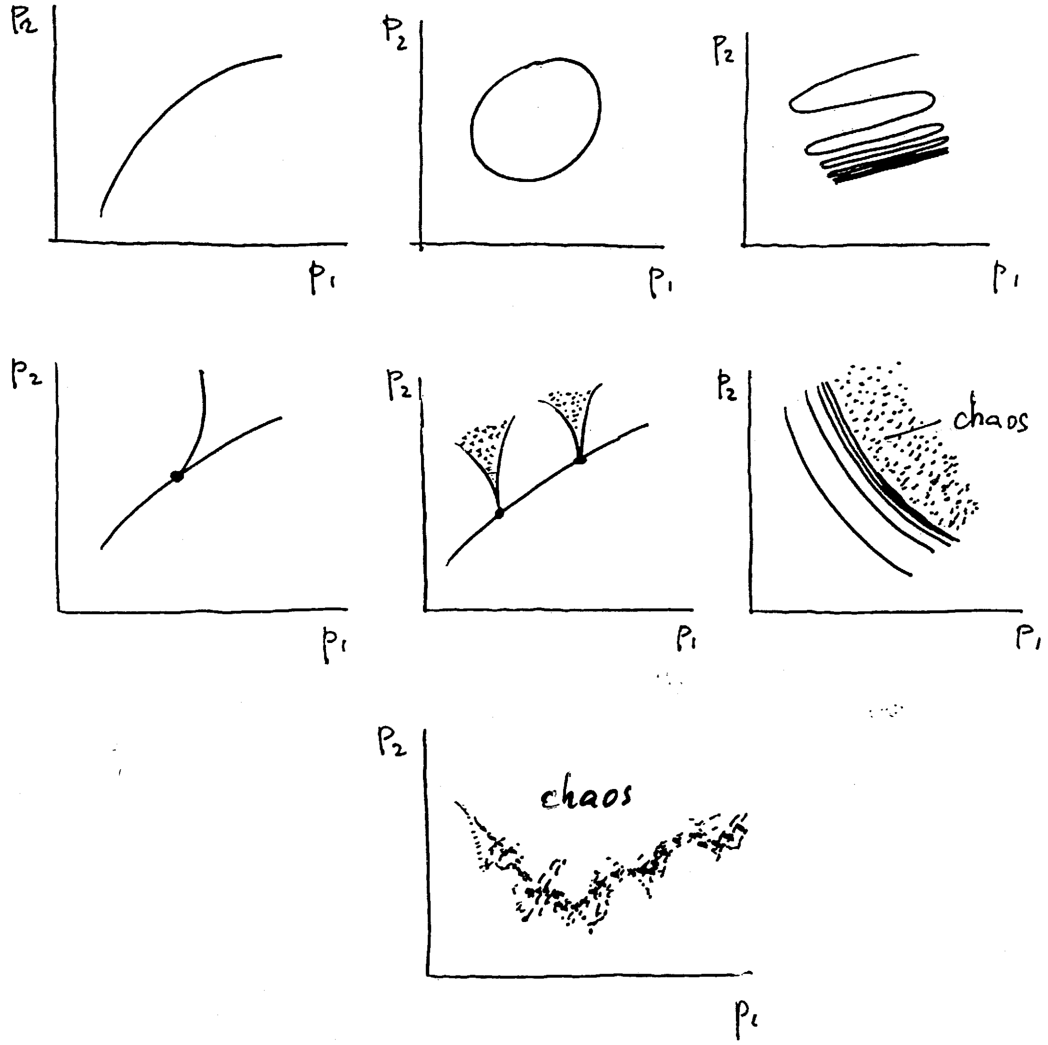


prey - predator



hydrodynamics

GEOMETRY OF BIFURCATION CURVES



- The regular bifurcation curves can be obtained by means of specialized software (continuation techniques) [AUTO - LOCBIF]

- 3D visualisation is particularly important

- chaos $\Rightarrow$ $\left\{ \begin{array}{l} \infty \text{ regions (i)} \\ \text{fractal regions (i)} \end{array} \right.$ $\Rightarrow$ very high complexity

CONCLUSIONS

physics

biology

economics $\longrightarrow$ $\dot{x} = f(x\ p)$ $\longrightarrow$ bifurcations

engineering

The analysis requires

- systems theory
- signal processing
- continuation techniques
- advanced software
- intuition

Existence of chaos $\Longrightarrow$ high complexity