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Optimal Resource Allocation (2)

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Part 2-1:

A few useful facts from mathematical analysis

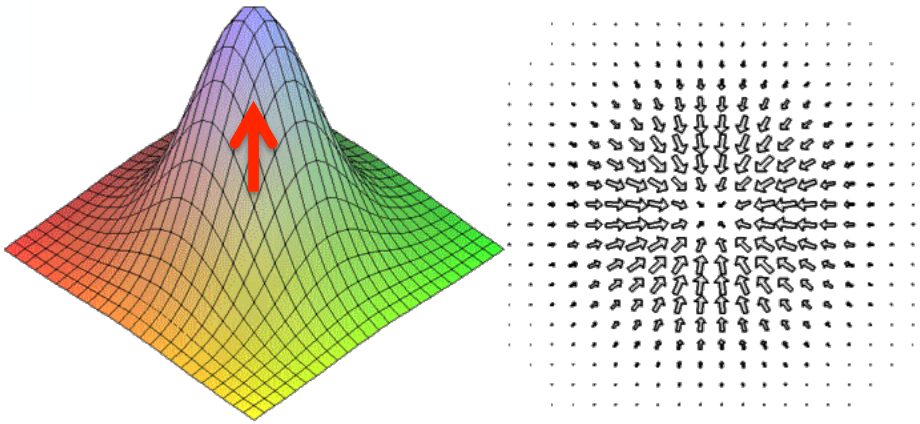
Gradient of a function

$$F(x_1,x_2)$$

$$grad \quad F(x_1,x_2)=\left(\frac{\partial F(x_1,x_2)}{\partial x_1},\frac{\partial F(x_1,x_2)}{\partial x_2}\right)$$

$$\frac{\partial F(x_1,x_2)}{\partial x_1}=\lim_{\Delta x_1\to 0}\frac{F(x_1+\Delta x_1,x_2)-F(x_1,x_2)}{\Delta x_1}$$

$$\frac{\partial F(x_1,x_2)}{\partial x_2}=\lim_{\Delta x_2\to 0}\frac{F(x_1,x_2+\Delta x_2)-F(x_1,x_2)}{\Delta x_2}$$



$grad \quad F(x_1,x_2)$ points in the direction of the greatest rate of increase of the function

Gradient of a linear function

$$F(x_1, x_2) = c_1 x_1 + c_2 x_2$$

$$grad \quad F(x_1, x_2) = (c_1, c_2)$$

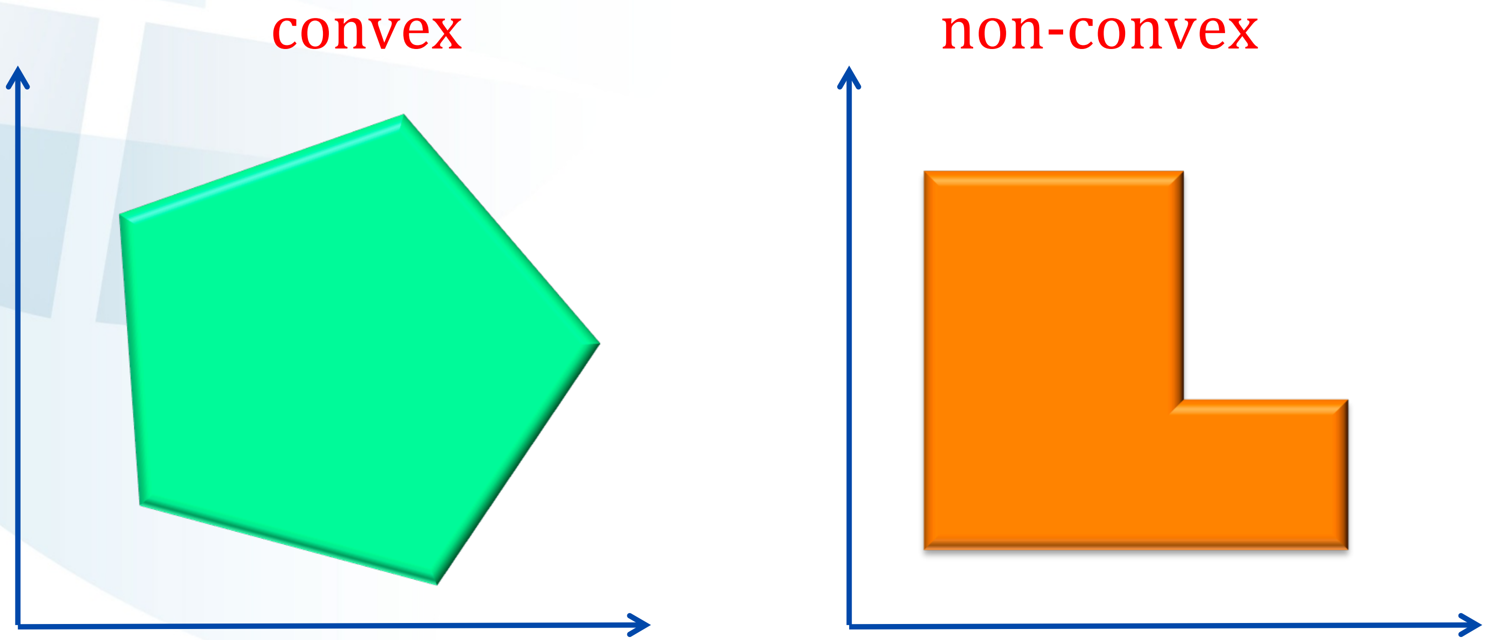
$$F(x) = cx \qquad c \in R^N \qquad x \in R^N$$

$$grad \quad F(x) = c$$

Function is a scalar, but the gradient is a vector!

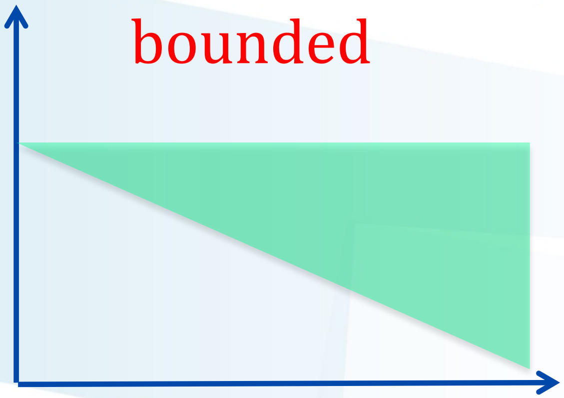
Convex sets

A **convex set** is a set where for every pair of points A and B from this set, the entire straight line that joins A and B belongs to this set too

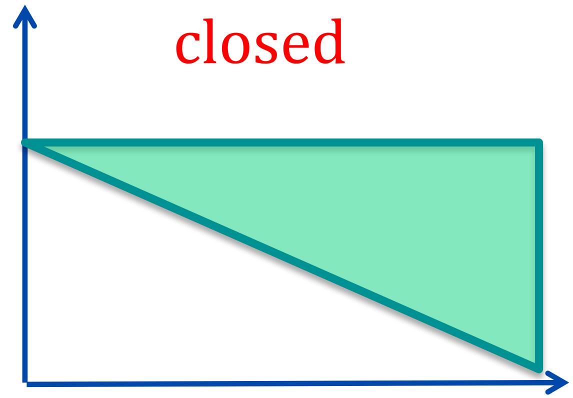


Boundedness and closedness

bounded

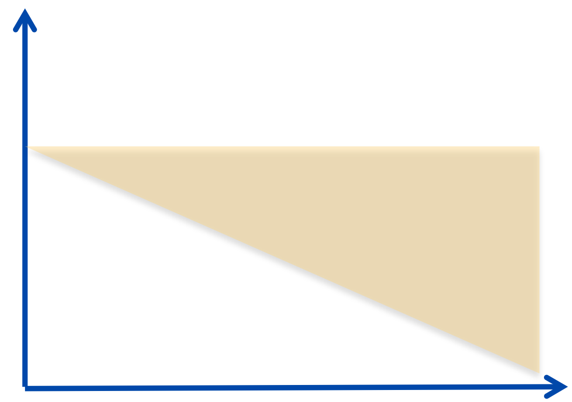


closed



unbounded

open



Rigorously defined in metric spaces

Rigorously defined by means of limit points

Questions?

Part 2-2:

Mathematical formalism of LP problems

Canonical form of a linear programming problem

$$cx \to \max$$
$$Ax \leq b$$
$$x \geq 0$$

$$c \in R^N$$
$$x \in R^N$$
$$b \in R^M$$
$$A : N \times M$$

A two-crop example

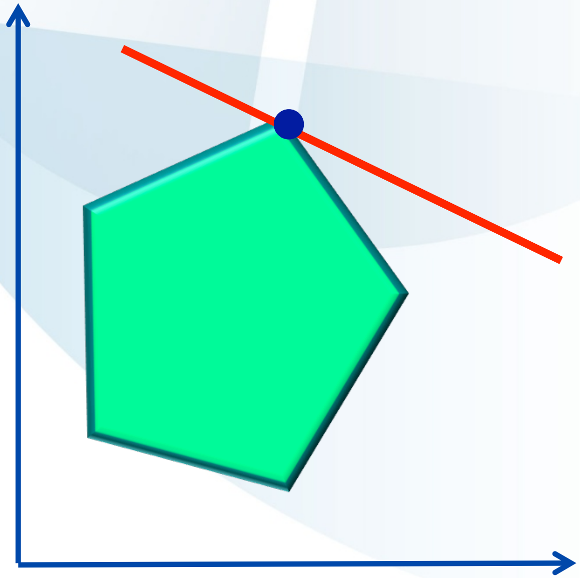
$$c_A x_A + c_B x_B \to \min$$
$$w_A x_A + w_B x_B \leq w$$
$$x_A + x_B \geq D$$
$$x_A \geq 0$$
$$x_B \geq 0$$

Exercise your understanding

- Write the two-crop example in vector-matrix form

Existence of a solution

$$cx \to \max$$
$$Ax \le b$$
$$x \ge 0$$



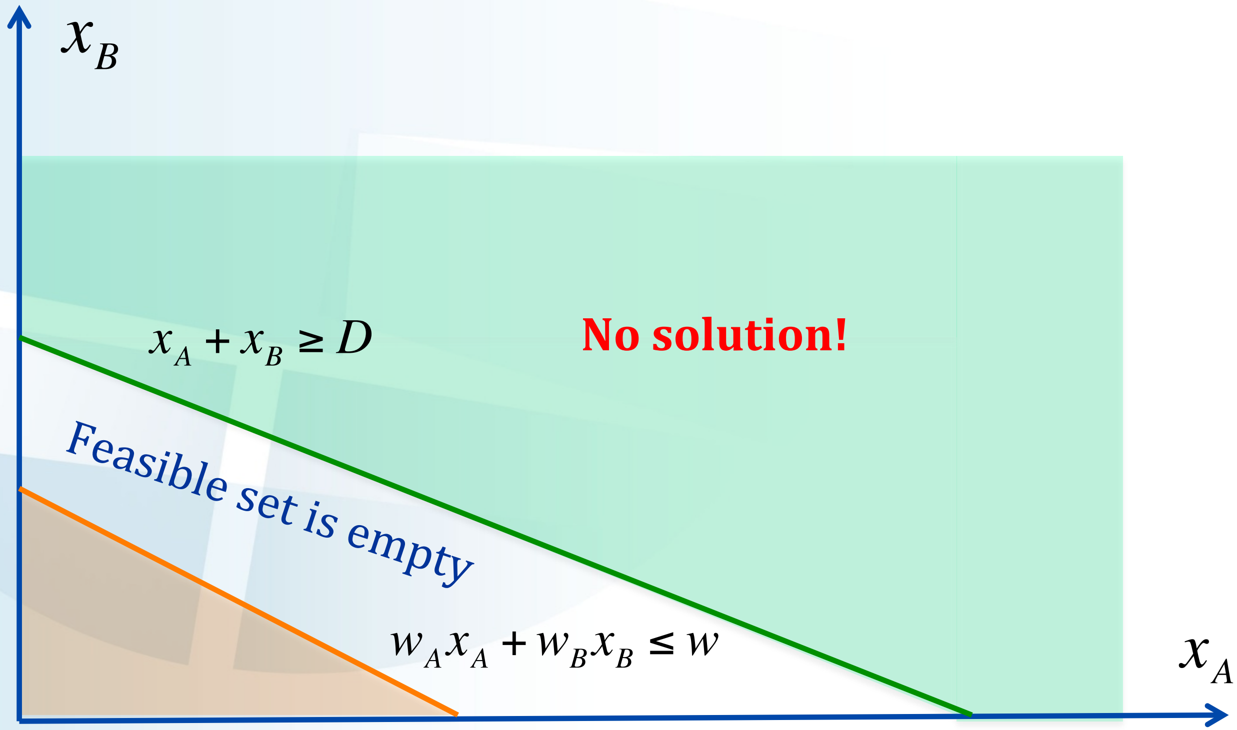
Key assumption:

$$X = \left\{ x \in R^N : Ax \le b \cap x \ge 0 \right\}$$

is non-empty and bounded in the direction of the gradient of the objective function c

=> The feasible set is a **convex polyhedron**
=> Linear function reaches its global minimum over a convex set => **Solution** to the LP problem **exists**

Infeasible constraints - example

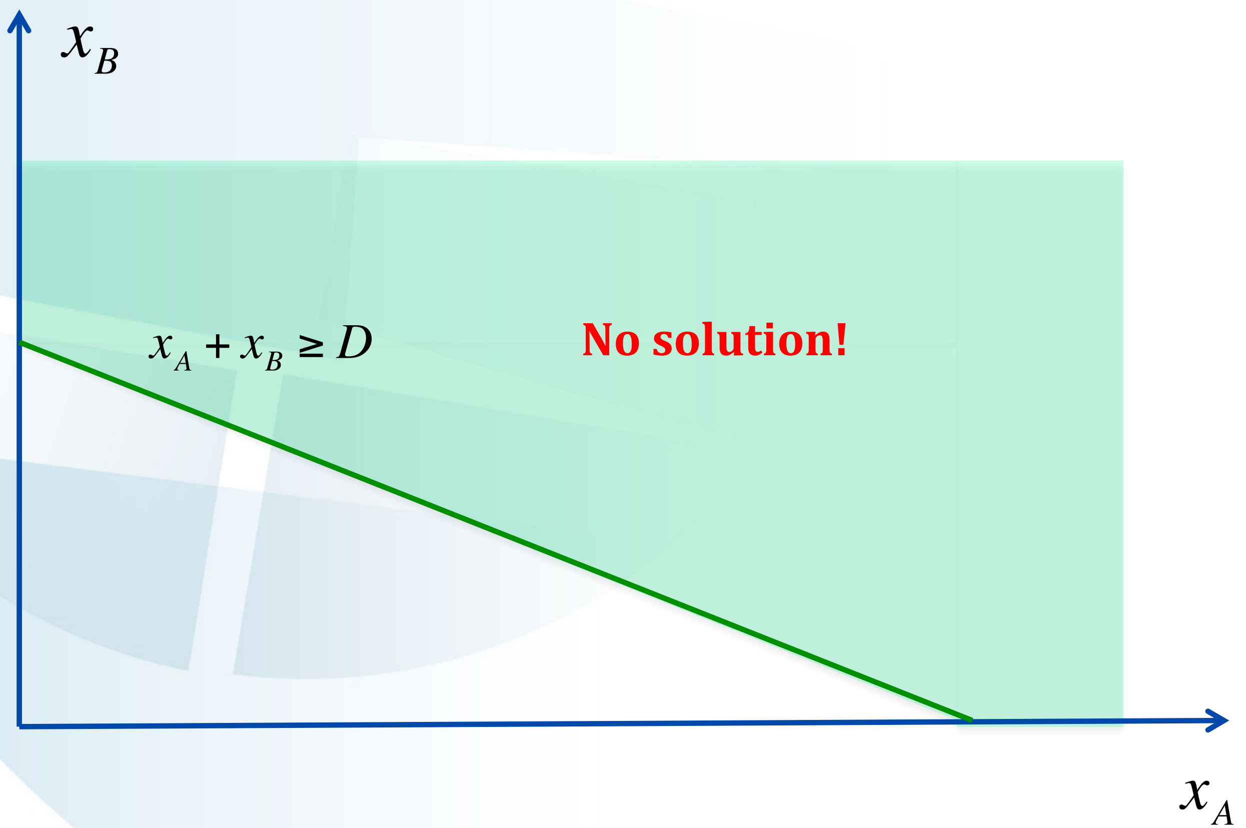


A two-crop example

$$c_A x_A + c_B x_B \rightarrow \min$$
$$w_A x_A + w_B x_B \leq w$$
$$x_A + x_B \geq D$$
$$x_A \geq 0$$
$$x_B \geq 0$$

Not possible to detect such a problem visually!
Important to set the constraints right!

Unboundedness - example



A two-crop example

$$p_A x_A + p_B x_B \rightarrow \max$$

$$\cancel{w_A x_A + w_B x_B \leq w}$$

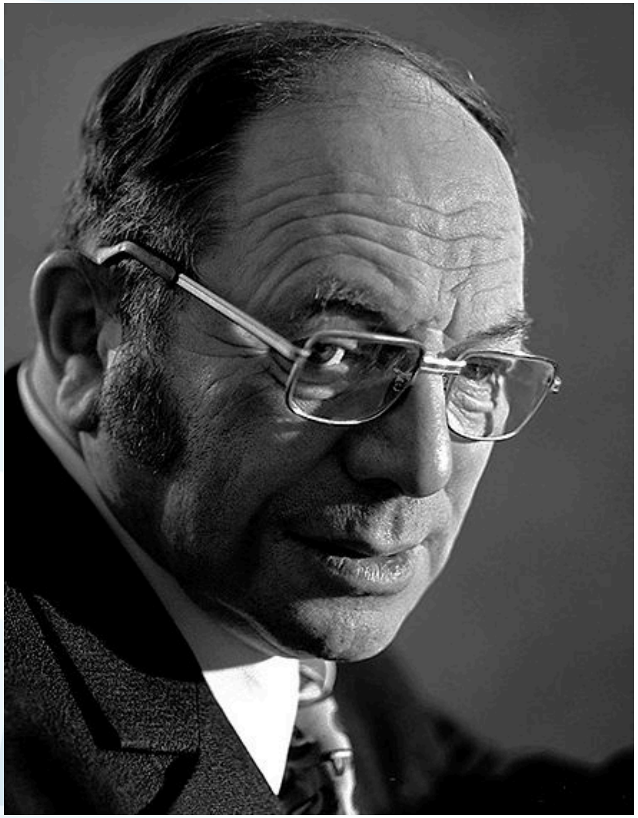
$$x_A + x_B \geq D$$

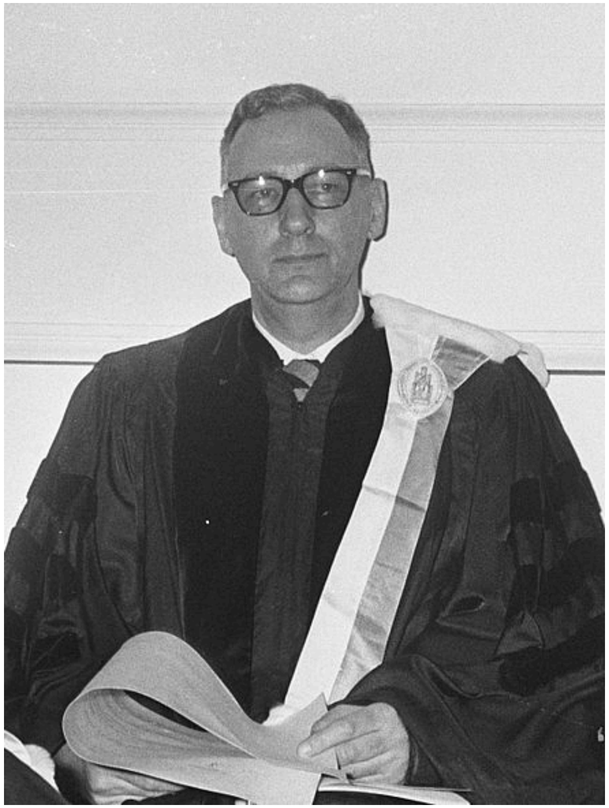
$$x_A \geq 0$$

$$x_B \geq 0$$

Not possible to detect such a problem visually!
Important to set the constraints right!

Leonid Kantorovich and TC Koopmanns



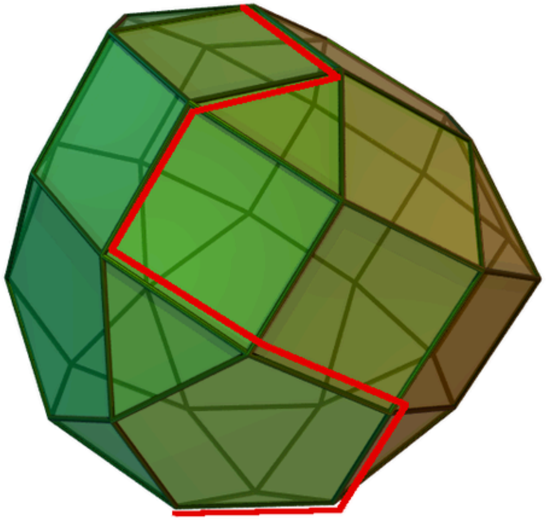


Nobel Prize 1975 winners for their contribution to the field of optimal resource allocation
- both were affiliated with IIASA in 1970s

George Dantzig

Formulated the “simplex” method to solve LP problems
- was also involved with IIASA in 1970s

The simplex algorithm is searching over edges of the polyhedron in the direction of the improvement of the objective function



Dual problem

Primal problem	Dual problem
$cx \to \max$	$by \to \min$
$Ax \le b$	$A^T y \ge c$
$x \ge 0$	$y \ge 0$

Strong duality: $cx^* = by^* = J^*$

A dual variable as a "shadow" price: $y_i^* = \dfrac{\Delta J^*}{\Delta b_i}$ $\left(\dfrac{\Delta J^*}{\Delta b} = \dfrac{\Delta b y^* + b\cancelto{0}{\Delta y^*}}{\Delta b} = y^* \right)$ 0 for small Δb

Exercise your understanding

- Write the dual problem, derive shadow prices and come up with their interpretation in the two-crop example

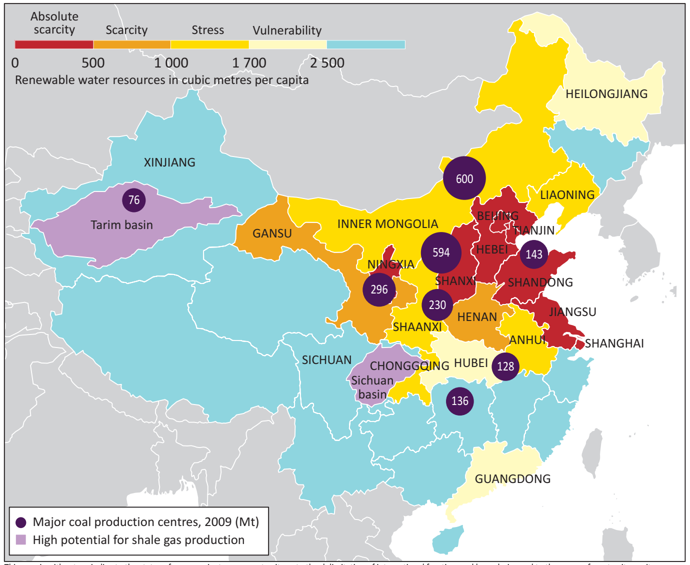
Questions?

Part 2-3:

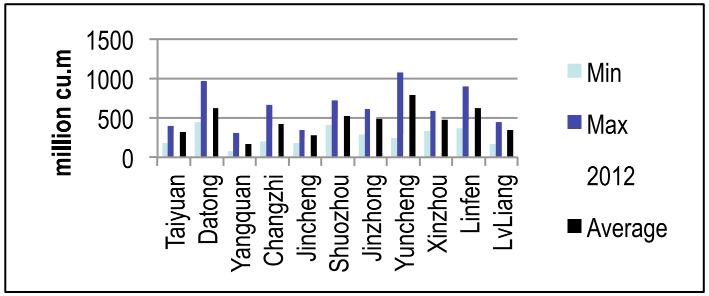
Example of application: Optimal land and water allocation between agriculture and coal mining in Shanxi, China

Motivation

- Coal is a major element of the energy security in China
- Coal mining tends to concentrate in water scarce regions, Shanxi province is a profound example
- Shanxi province is rich in coal (40% of the national reserve; produces 25% of total coal in China)
- Coal-bearing area occupies ~40% of the total area
- Only 30% of the arable land is irrigated, yields largely depend on rainfalls
- ~30% of basic food is imported from other provinces
- Coal mining and arable land overlap by up to 40%



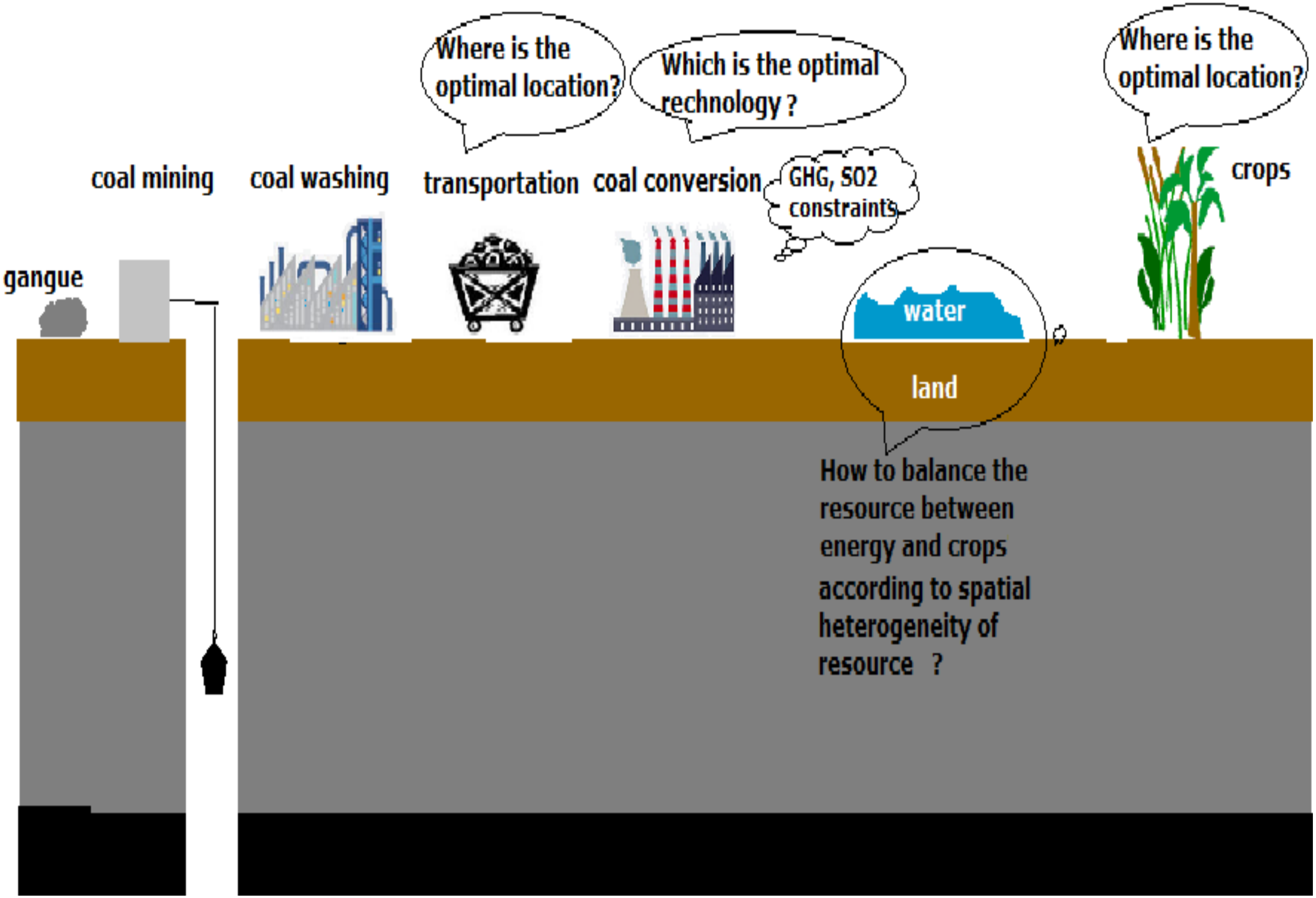
This map is without prejudice to the status of or sovereignty over any territory, to the delimitation of international frontiers and boundaries and to the name of any territory, city or area.



Water availability across Shanxi Province in 1994-2012

Strong competition between agrifood production and coal production for land and water

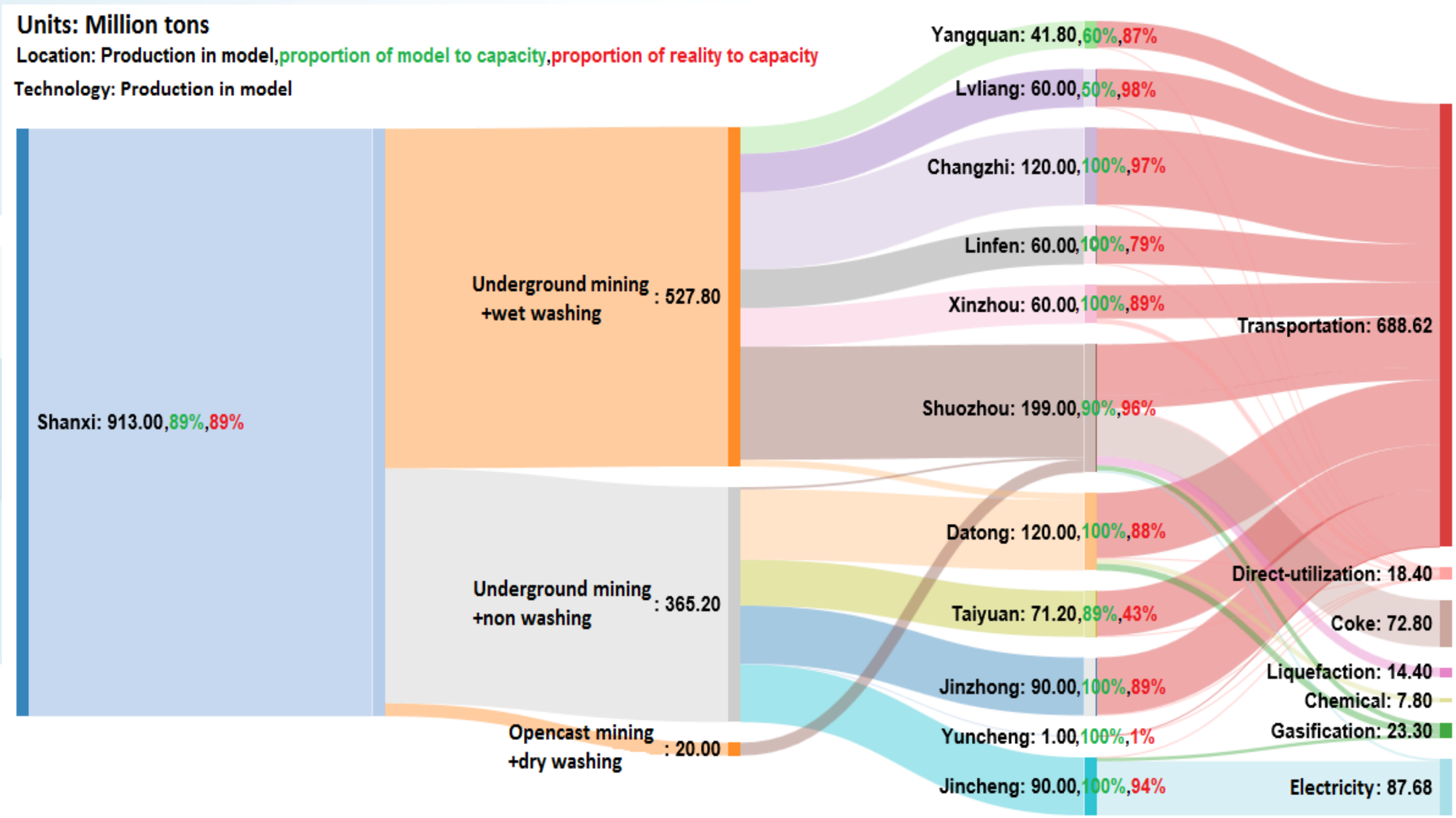
Model sketch



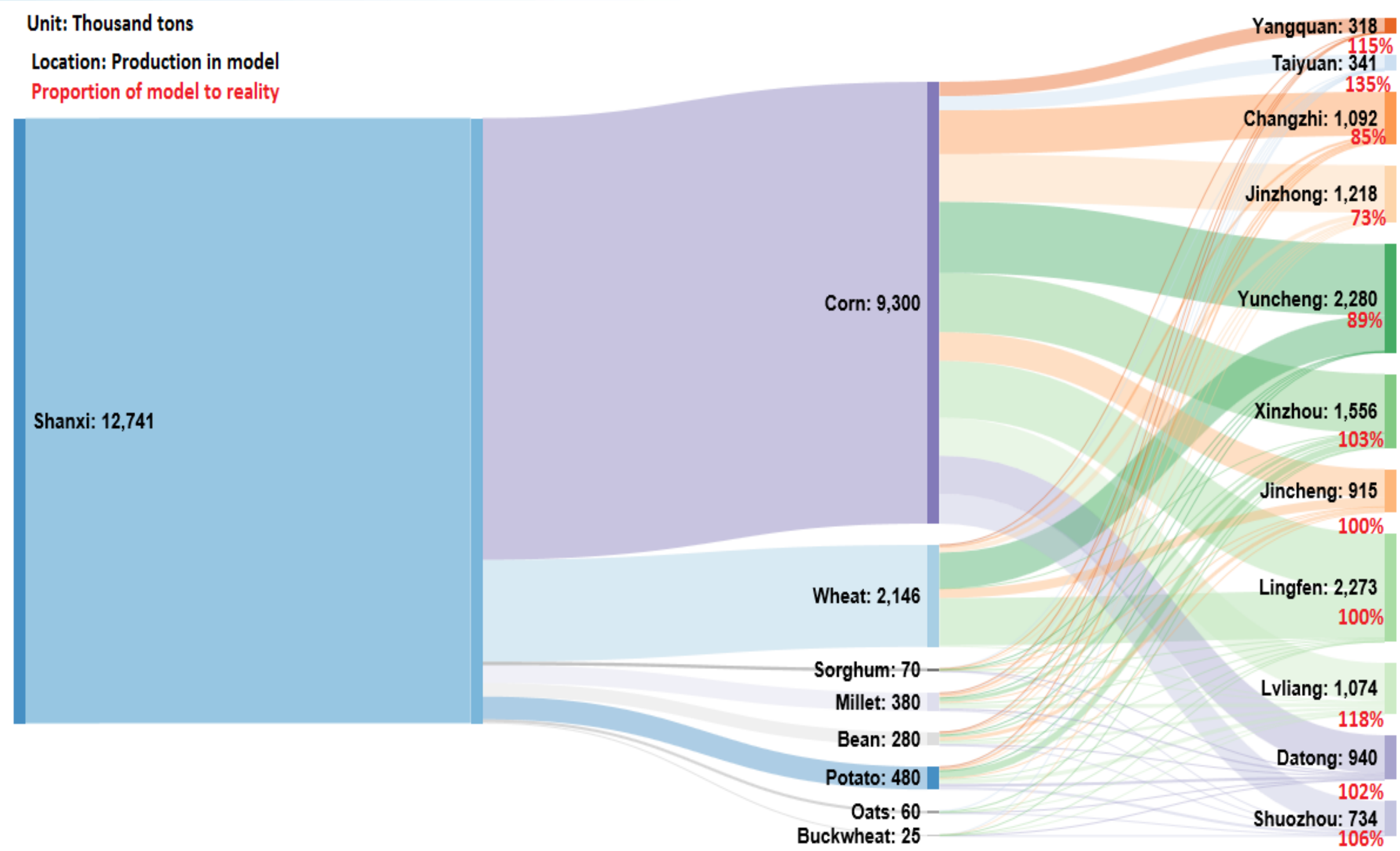
Optimal resource allocation model

- Calibrated based on 2012 data
- Redistributes production of major crops and coal across 11 prefectual cities
- Minimizes the total costs, including transportation between cities
- Illuminates and quantifies the tradeoffs between the coal and agriculture sectors
- Analyzes the dependence of an optimal solution to the water availability scenario
- Estimates the shadow prices

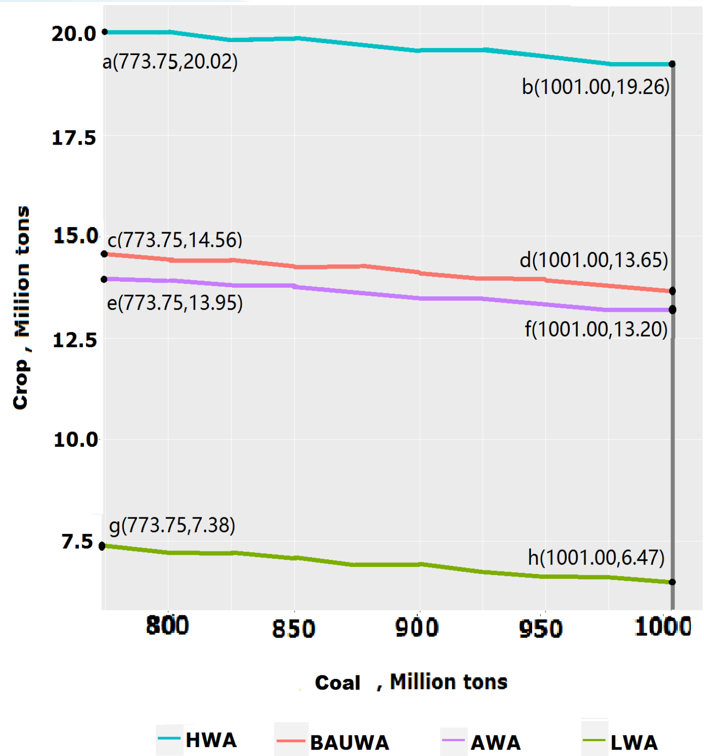
Modeling results: Redistribution of coal production

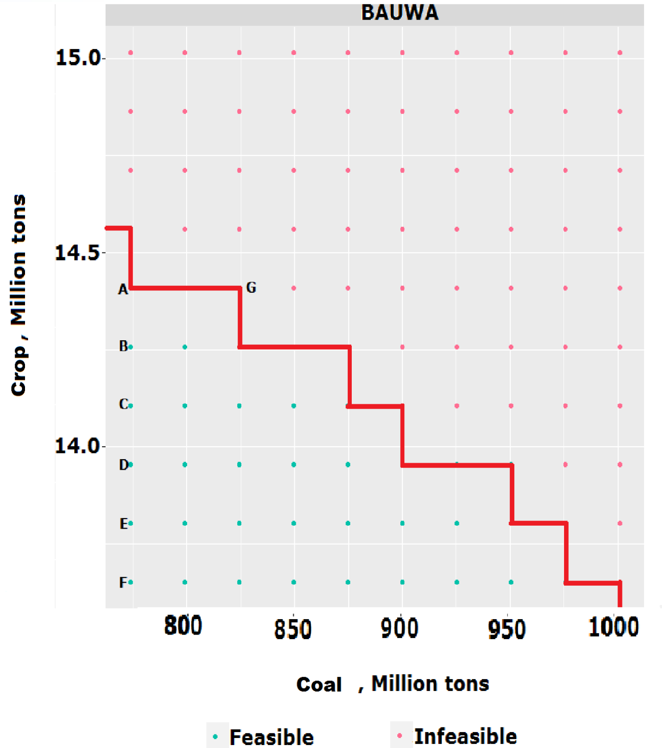


Modeling results: Redistribution of crop production

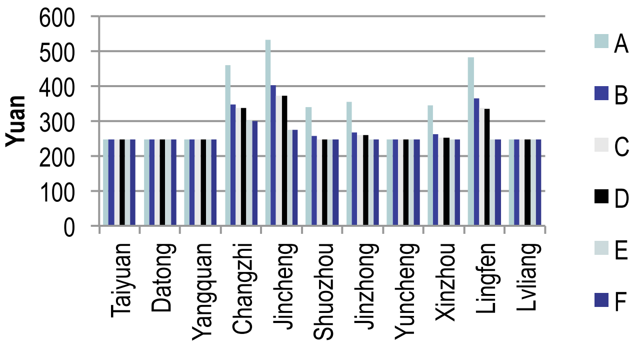


Modeling results: Tradeoffs and sensitivity to water availability





HWA/LWA/AWA assumes the maximal/ minimal/average observed water availability in each city over 1994-2012



Shadow prices

Questions?