ON THE VALUE OF POPULATION IN DISTRIBUTED CONTROL MODELS

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IIASA-Seminar, Advanced Systems Analysis Group

1 February 2017

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Motivation I

- Research and management regularly assign values to populations as well as to individual members of a population:

- Demography / evolutionary biology: Reproductive value, evolutionary fitness
- Resource management: Hotelling rule, Hartwick rule, etc. assign values (e.g. of instantaneous harvest as opposed to future harvest)
- Environmental management (i): use value + option value + non-use value
- Environmental management (ii): Environmental impact of humans (e.g. carbon footprint)
- Business management: capital stock, inventory

Motivation 2

Valuing human life:

“Economists know the price of everything but the value of nothing” (Oscar Wilde)

- Value of a statistical life in medicine, health and environmental economics, occupational safety, public policy
- Human capital
- Valuing birth control (in terms of development impacts)
- Valuing environmental impact of a person (e.g. carbon footprint)

Motivation 3

Conceptually:

- Valuations can be implicit (shadow price of population stock) or…
- …explicit, e.g. the reproductive value (measured in numbers of offspring) or the statistical value of life (measured in money)

Our contribution (Wrzaczek et al. , Theoretical Population Biology, 2010):

- Formal link between implicit valuations in distributed control models and (economic) valuations
- Applications to health economics, epidemiology, resource management in a predator-prey-setting.

Basics

Population decrement (McKendrick equation):

$$N_a + N_t = -\mu(a,t)N(a,t) \qquad N(0,t) = B(t), N(a,0) = N_0(a)$$

Population renewal:

$$B(t) = \int_0^\omega \underbrace{\nu(a,t)}_{\text{fertility at age s}} N(a,t)\,da$$

(Remaining) Reproductive value (Fisher, 1930):

$$\psi^R(a,t) = \int_a^\omega e^{-n(s-a)} \underbrace{\frac{l(s,t-a+s)}{l(a,t)}}_{\text{survival from a to s}} \nu(s,t-a+s)ds$$

Rate of population growth

Application 1: Survival investments vs. consumption

- Planner maximizes social welfare of an aged structured population over time by choosing consumption c(a,t) and survival investments h(a,t)

- Optimal allocation rule for h(a,t) embraces the statistical value of life amended by a value of progeny (Kuhn et al. 2010)

State dynamics:

Population $$\left(\frac{\partial}{\partial a}+\frac{\partial}{\partial t}\right)N(a,t) = -\mu(a,h(a,t))N(a,t) \qquad N(0,t)=B(t), N(a,0)=N_0(a)$$

Wealth $$\left(\frac{\partial}{\partial a}+\frac{\partial}{\partial t}\right)A(a,t) = rA(a,t) + (\underbrace{y(a)}_{\text{exogenous income}} - c(a,t) - h(a,t))N(a,t)$$

$$A(0,t) = A(\omega,t) = 0 \quad \forall t$$

interest rate

$$A(a,t) = A_0(a), A(a,T) = A_T(a) \quad \forall a,$$

Application 1: Survival investments vs. consumption

Social welfare objective:

rate of time preference

$$\max_{c(a,t),h(a,t)} \int_0^T \int_0^\omega e^{-\rho t} \underbrace{u(c(a,t))}_{\text{flow utility}} N(a,t) \; da \; dt$$

s.t. population and wealth dynamics + boundary conditions

Value of the population state:

Direct „value" Productive value

$$\begin{aligned} \xi^N(a,t) \;\; &= \;\; \int_a^\omega e^{-\rho(s-a)} \frac{l(s,t-a+s)}{l(a,t)} \Big[u(c) + u_c(c)(y-c-h) \Big] \; ds \\ &+ \underbrace{\int_a^\omega e^{-\rho(s-a)} \frac{l(s,t-a+s)}{l(a,t)} \nu(s,t-a+s) \xi^N(0,t-a+s) ds}_{\text{(Remaining) reproductive value}} \end{aligned}$$

Application 1: Survival investments vs. consumption

First-order conditions on controls:

$$\mathcal{H}_c = u_c(c)N - \xi^A N = 0$$

consumption smoothing across age-groups

$$\mathcal{H}_h = -\xi^N \mu_h(a,h)N - \xi^A N = 0$$

optimal survival investments

or in economic terms:

$$\underbrace{-\frac{1}{\mu_h(a,h)}}_{\text{effective \$ cost of saving one life}} = \frac{\xi^N}{\xi^A} = \underbrace{\psi^N(a,t)}_{\text{monetary value of a life saved}}$$

Application 1: Survival investments vs. consumption

Monetary value of survival

$$\psi^N(a,t) = \overbrace{\int_a^{\omega} e^{-r(s-a)} \frac{l(s,t-a+s)}{l(a,t)} \left[\frac{u(c)}{u_c(c)} + (y-c-h) \right] ds}^{\text{\$ Value of a statistical life (as e.g. in Shepard \& Zeckhauser, 1984, MSI}}$$

$$+ \int_a^{\omega} e^{-r(s-a)} \frac{l(s,t-a+s)}{l(a,t)} \nu(s,t-a+s) \frac{u_c(c(0,t-a+s))}{u_c(c(s,t-a+s))} \underbrace{\psi^N(0,t-a+s)ds}_{\text{\$ value of a birth at age s}}$$

Consumer surplus
=$ value of utility

Conversion from „newborn“ values into survivor‘s values (at age s)

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Application 1: Survival investments vs. consumption

Value of a birth (value of progeny)

$$\psi^N(0,t) = \underbrace{\int_0^\omega e^{-rs} l(s,t+s)\left[\frac{u(c)}{u_c(c)}\right]ds}_{\text{\$ value of a full life lived}} + \underbrace{\int_0^\omega e^{-rs} l(s,t+s)\nu(s,t+s)\frac{u_c(c(0,t+s))}{u_c(c(s,t+s))}\psi^N(0,t+s)ds}_{\text{\$ value of offspring}}$$

Note: zero productive value if life-cycle budget is balanced

For a stable population & steady state economy:

Implies and is implied by dynamic efficiency

$$\psi^N(0,t) = \frac{\int_0^\omega e^{-rs} l(s,t+s)\left[\frac{u(c)}{u_c(c)}\right]ds}{\int_0^\omega \left[e^{(r-n)s}-1\right] e^{-rs} l(s,t+s)\nu(s,t-a+s)ds} \in [0,\infty) \iff r > n$$

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Application II: Management of a predator-prey system

- Two interacting animal populations: predators (=wolves) and prey (=sheep)
- Optimal culling policies u(a,t) aimed at wolves and w(a,t) aimed at sheep
- Predator-prey interaction driven by age-specific hunting effectiveness f(a) and vulnerability g(a)…
- …translating into:

$$P(t) = \int_0^\omega g(a)B(a,t)da$$ Pool of „effective prey“

$$Q(t) = \int_0^\omega f(a)R(a,t)da$$ Predatory pressure

Application II: Management of a predator-prey system

Population dynamics

- **Wolves:**

$$\begin{aligned}\left(\frac{\partial}{\partial a}+\frac{\partial}{\partial t}\right)R(a,t) &= \overbrace{-\mu_R(a)R(a,t)}^{\text{natural mortality}} - \overbrace{h_R(a,u(a,t))}^{\text{human culling at effort u(a,t)}} \\ R(0,t) &= G(t) = \int_0^{\omega} \underbrace{\nu_R(a)R(a,t)f(a)P(t)}_{\text{Surviving offspring for a mom aged a increases in her hunting success f(a)P(t)}}\ da,\ R(a,0) = R_0(a)\end{aligned}$$

- **Sheep:**

$$\begin{aligned}\left(\frac{\partial}{\partial a}+\frac{\partial}{\partial t}\right)B(a,t) &= \overbrace{-\mu_B(a)B(a,t)g(a)Q(t)}^{\text{natural + wolve-induced mortality depending on „predatory risk“ g(a)Q(t)}} - \overbrace{h_B(a,w(a,t))}^{\text{human culling at effort w(a,t)}} \\ B(0,t) &= H(t) = \int_0^{\omega} \nu_B(a)B(a,t)da,\ B(a,0) = B_0(a)\end{aligned}$$

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Application II: Management of a predator-prey system

Objective: Net present $ value of the system

$$\max_{u(a,t),w(a,t)} \int_0^T \int_0^\omega e^{-\rho t} F(R(a,t), B(a,t), u(a,t), w(a,t)) da dt$$

s.t. predator-prey dynamics + boundary conditions

First-order conditions on controls:

$$\mathcal{H}_u = F_u - \xi^R(a,t) \frac{\partial h_R(a, u(a,t))}{\partial u(a,t)} = 0$$

$$\mathcal{H}_w = \underbrace{F_w} - \xi^B(a,t) \frac{\partial h_B(a, w(a,t))}{\partial w(a,t)} = 0$$

marginal return (through sale of culled animals)
– marginal cost of culling

Application II: Management of a predator-prey system

Value of a predator (wolf) aged a at time t

$$\xi^R(a,t) = \int_a^\omega e^{-\int_a^s[\rho+\mu_R(s')]ds'} \Big(F_R(\cdot) - f(s)\int_0^\omega \mu_B(s')g(s')\,B(s',t-a+s)\xi^B(s',t-a+s)ds' \Big) ds + \int_a^\omega e^{-\int_a^s[\rho+\mu_R(s')]ds'} \xi^R(0,t-a+s)\nu_R(s)f(s)P(t-a+s)ds$$

Direct value of predator (eco-system value?) ≥0

Value of the predator‘s offspring: <0 for F_R=0

Loss of prey animals valued at ξ^B>0 due to increased predatory pressure: <0

Application II: Management of a predator-prey system

Value of a prey animal (sheep) aged a at time t

$$\begin{aligned}
\xi^B(a,t) \; &= \; \int_a^\omega e^{-\int_a^s [\rho+\mu_B(s')g(s')Q(t-a+s')]ds'} \Big(F_B(\cdot) \\
&\quad + g(s) \int_0^\omega \nu_R(s') f\,(s')\, R(s', t-a+s)\xi^R(0,t-a+s)ds' \Big) ds \\
&\quad + \int_a^\omega e^{-\int_a^s [\rho+\mu_B(s')g(s')Q(t-a+s')]ds'} \xi^B(0,t-a+s)\nu_B(s)ds
\end{aligned}$$

Direct use value of prey (e.g. value of wool, milk, etc.) + option value of future cull >0

Value of the prey's offspring: >0 for F_B>0 (suff. large)

Support of predator offspring valued at ξ^R<0 due to greater pool of prey: <0

Application III: HIV prevention

- Two populations: susceptibles (S) and infected (I)

- Susceptible age group S(a,t) interacts with infected at an age-specific rate

$$P(a,t) = \int_0^{\omega} \lambda(a,a') \frac{I(a',t)}{S(a',t)+I(a',t)} \; da'$$

- with γ(a) = base risk of infection and ϕ(u(a,t))≤1 the impact of an prevention effort u(a,t) the number of newly infected amongst age group a at time t is given by

$$\gamma(a)\phi(u(a,t))P(a,t)S(a,t)$$

Application III: HIV prevention

Population dynamics

- **Susceptibles:**

$$\left(\frac{\partial}{\partial a}+\frac{\partial}{\partial t}\right)S(a,t) = \overbrace{-\mu_S(a)S(a,t)}^{\text{Natural deaths}} - \overbrace{\gamma(a)\phi(u(a,t))P(a,t)S(a,t)}^{\text{New infections}}$$

$$S(0,t) = \underbrace{B(t)}_{\text{Births by susceptibles}} + \underbrace{\alpha C(t)}_{\text{Non-infected births by the infected}},\ S(a,0) = S_0(a)$$

- **Infected:**

$$\left(\frac{\partial}{\partial a}+\frac{\partial}{\partial t}\right)I(a,t) = \overbrace{-\mu_I(a)I(a,t)}^{\text{Deaths from infection (or other)}} + \overbrace{\gamma(a)\phi(u(a,t))P(a,t)S(a,t)}^{\text{New infections}}$$

$$I(0,t) = \underbrace{(1-\alpha)C(t)}_{\text{Infected births by the infected}},\ I(a,0) = I_0(a)$$

- with $B(t) = \int_0^{\omega} \nu_S(a,t)S(a,t)da$

$$C(t) = \int_0^{\omega} \nu_I(a,t)I(a,t)da$$

Application III: HIV prevention

Objective: Minimise the total cost of infection plus prevention

$$\max_{u(a,t)} - \int_0^T \int_0^\omega e^{-\rho t} \Big(\underbrace{F(I(a,t))}_{\text{Cost of infection related to age group a at time t}} + \underbrace{u(a,t)}_{\text{Prevention cost related to age group a at time t}} \Big) dadt$$

First-order condition on control:

$$\mathcal{H}_u = \underbrace{-1}_{\text{Marginal \$ spent on prevention}} - \underbrace{\gamma(a)\, P(t)\, S(a,t)\, \frac{\partial \phi}{\partial u}}_{\text{Marginal number of infections prevented}} \underbrace{\left[\xi^S(a,t) - \xi^I(a,t) \right]}_{\text{Value of averting an infection}} = 0$$

Application III: HIV prevention

Value of a susceptible (in terms of HIV-related costs)

$$
\begin{aligned}
\xi^S(a,t) &= \int_a^\omega e^{-\int_a^s \left[\rho+\mu_S(s')+\gamma\phi(u)P\right]ds'} \Big[\overbrace{\gamma\phi(u)P\xi^I}^{\text{Expected value of becoming infected }(<0)} \overbrace{- \int_0^\omega \zeta(t,a')\frac{\lambda I}{(S+I)^2}da'}^{\text{Value in lowering disease prevalence and infection risk: typically }>0} \Big] ds \\
&+ \int_a^\omega \underbrace{e^{-\int_a^s \left[\rho+\mu_S(s')+\gamma\phi(u)P\right]ds'} \xi^S(0,t-a+s)\nu_S(s)ds}_{\text{Value of non-infected children}}
\end{aligned}
$$

Application III: HIV prevention

Value of an infected (in terms of HIV-related costs)

$$\xi^I(a,t) = \int_a^\omega e^{-\int_a^s[\rho+\mu_I(s')]ds'}\Big[\overbrace{-F_I(I)}^{\text{Contribution of an additional infected to the cost of infection}} + \overbrace{\int_0^\omega \zeta(t,a')\frac{\lambda S}{(S+I)^2}da'}^{\text{Value of rasing disease prevalence and infection risk: typically <0}}\Big]ds$$

$$+\int_a^\omega e^{-\int_a^s[\rho+\mu_I(s')]ds'}\Big(\underbrace{\alpha\xi^S(0,t-a+s)}_{\text{Expected value of non-infected children}} + \underbrace{(1-\alpha)\xi^I(0,t-a+s)}_{\text{Expected value of infected children <0 if }\alpha\to 0}\Big)\nu_I(s)ds$$

Conclusion

- Shadow prices on population states can be interpreted as values of population. These could be directly in \$ terms if the objective function is in \$, or they could be converted into \$ terms, als in case of the value of human survival.
- As such they can be used to

  (a) compare the valuations in the model against the data (to check e.g. if components of the model are missing)
  (b) to calibrate the model to the data
  (c) to provide policy guidance.
- An analysis of the components of the shadow price allows to identify components relevant to the valuation (and the mechanisms behind the model)
- In age-structured models of long-term development of the population, the value of population falls into a „direct“ part relating to the contemporary individual/cohort, and an indirect relating to the value of the individuals expected offspring
- The value of population is age-specific in a non-trivial way

Appendix: General model

Salvage value related to non-pop states

$$\max_{u,v,w} \quad \int_0^T \int_0^\omega e^{-\rho t} L(a,t,N,Y,Q,P,u,v,w)\ da\ dt + \overbrace{\int_0^\omega e^{-\rho T} l(a, Y(a,t))\ da}$$

$$\text{s.t.} \quad \left(\frac{\partial}{\partial a} + \frac{\partial}{\partial t}\right) N(a,t) = -\mu(a,t,N,Y,Q,P,u)N + g(a,t,N,Y,Q,P,u)$$

Other population change

$$N(0,t) = B(t) = \int_0^\omega \nu(a,t,N,Y,Q,P,u)N\ da, \quad N(a,0) = N_0(a)$$

Density dependent mortality and fertility

$$\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial t}\right) Y(a,t) = f(a,t,N,Y,Q,P,u)$$

$$Y(0,t) = \varphi(t,B,Q,v), \quad Y(a,0) = Y_0(a,w)$$

Non-population states

$$Q(t) = \int_0^\omega h(a,t,N,Y,Q,P,u)\ da$$

Boundary states

$$P(a,t) = \int_0^\omega k(a,t,a',N,Y,u)\ da'$$

Interaction states

$$\underbrace{u(a,t)}_{\text{Age-structured controls}} \in U, \quad \underbrace{v(t)}_{\text{Boundary controls}} \in V, \quad \underbrace{w(a)}_{\text{Initial controls}} \in W$$

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Appendix: General model

The distributed, initial and boundary Hamiltonian of the general model reads as follows

$$
\begin{aligned}
\mathcal{H}(\cdot) &= L + \xi^N(-\mu N + g) + \xi^Y f + \eta^B \nu N + \eta^Q h + \int_0^\omega \zeta k(a,t,a',u)\ da' \\
\mathcal{H}_0(\cdot) &= \xi^N(a,0)N_0(a) + \xi^Y(a,0)Y_0(a,w) + \int_0^T L(a,t,w)\ dt \\
\mathcal{H}_b(\cdot) &= \xi^N(0,t)B(t) + \xi^Y(0,t)\varphi(t,v) + \int_0^\omega L(a,t,v)\ da. \qquad (1)
\end{aligned}
$$

Applying distributed optimal control theory (see Feichtinger et al. (2003)) we obtain the following adjoint system

$$
\begin{aligned}
\xi_a^N + \xi_t^N &= (\rho + \mu + \mu_N N)\xi^N - L_N - \xi^N g_N - \xi^Y f_N - \eta^B(\nu_N N + \nu) - \eta^Q h_N - \int_0^\omega \zeta^P k_N\ da' \\
\xi_a^Y + \xi_t^Y &= (\rho + f_Y)\xi^Y - L_Y + \xi^N \mu_Y N - \xi^N g_Y - \eta^B \nu_Y N - \eta^Q h_Y - \int_0^\omega \zeta^P k_Y\ da' \\
\eta^B &= \xi^N(0,t) + \xi^Y(0,t)\varphi_B \\
\eta^Q &= \xi^Y(0,t)\varphi_Q + \int_0^\omega L_Q + \xi^N(-\mu_Q N + g_Q) + \xi^Y f_Q + \eta^B \nu_Q N + \eta^Q h_Q\ da \\
\zeta^P &= L_P + \xi^N(-\mu_P N + g_P) + \xi^Y f_P + \eta^B \nu_P N + \eta^Q h_P
\end{aligned}
$$

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Appendix: General model

together with the transversality conditions

$$\xi^N(a,T) = 0 \qquad \xi^N(\omega,t) = 0$$
$$\xi^Y(a,T) = l_Y(a,T) \qquad \xi^Y(\omega,t) = 0$$

Finally the necessary first order conditions can be derived from

$$\begin{aligned}
\mathcal{H}(a,t,u^*(a,t)) &\geq \mathcal{H}(a,t,u(a,t)) && \forall\, u(a,t) \in U \\
\frac{\partial \mathcal{H}_0}{\partial w}(a_0, w^*(a_0))(w - w^*(a_0)) &\leq 0 && \forall\, w \in W \\
\frac{\partial \mathcal{H}_0}{\partial v}(t_0, v^*(t_0))(v - v^*(t_0)) &\leq 0 && \forall\, v \in V
\end{aligned}$$

where $u^*(a,t)$ denotes the distributed, $w^*(a_0)$ the initial and the $v^*(t_0)$ the boundary optimal control.

Appendix: General model

$$
\begin{aligned}
\Big(\frac{\partial}{\partial a}+\frac{\partial}{\partial t}\Big)\xi^N(a,t) &= (\rho+\mu+\mu_N N - g_N)\xi^N - L_N - \xi^Y f_N - \\
&\quad -\xi^N(0,t-a+s)(\nu_N N+\nu) - \eta^Q h_N - \int_0^{\omega} \zeta P_N \ da' \qquad (2)
\end{aligned}
$$

where $\xi^Y(a,t)$, $\eta^Q(t)$ and $\zeta(a,t)$ are the adjoint variables of Y, Q and P respectively. All adjoint variables can be interpreted as dynamic shadow prices, i.e. they indicate the increase of the objective function if the corresponding state is increased marginally. E.g. $\xi^N(a,t)$ denotes the increase of the objective function if the population is increased marginally at age a at time t (or by one a-aged individual at t if the population is large enough). The term shadow price has already been used in the examples.

Together with the transversality condition $\xi^N(\omega,t)=0$ the shadow price of the population can be solved with the method of characteristics for all cohorts whose maximal life horizon ends before the planning horizon T

$$
\begin{aligned}
\xi^N(a,t) &= \int_a^{\omega} e^{-\int_a^s(\rho+\mu+\mu_N N-g_N)ds'}\Big(L_N+\xi^Y f_N+\eta^Q h_N+\int_0^{\omega}\zeta P_N \ da'\Big) \ ds + \\
&\quad + \int_a^{\omega} e^{-\int_a^s(\rho+\mu+\mu_N N-g_N)ds'}\xi^N(0,t-a+s)(\nu+\nu_N N) \ ds. \qquad (3)
\end{aligned}
$$

Acknowledgment

THANK YOU!