

**OPTIMAL CONTROL OF A PERTURBED  
SWEEPING PROCESS**

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## THE CONTROLLED SWEEPING PROCESS

is described by the dissipative differential inclusion

$$\begin{cases} -\dot{x}(t) & \in N(x(t); C(t)) \text{ a.e. } t \in [0, T] \\ x(0) & = x_0 \in C(0), \end{cases}$$

where  $N(\cdot; \Omega)$  stands for the usual normal cone of convex analysis, and where  $t \mapsto C(t)$  is a Lipschitzian set-valued mapping (moving set). Classical theory of the sweeping process establishes the existence and **uniqueness** of Lipschitzian solutions for a **given** moving set  $C(t)$ , and so doesn't allow any room for optimization. We suggest **to control** the sweeping set  $C(t)$  by some forces and thus formulate and study new classes of **optimal control problems** for controlled sweeping process with various applications; in particular, to **quasistatic elastoplasticity**, **magnetic hysteresis**, **social-economic modeling**, etc.

## OPTIMAL CONTROL PROBLEM

Given a terminal cost function  $\varphi$  and a running cost  $\ell$ , consider the optimal control problem  $(P)$ : minimize

$$J[x, u, b] = \varphi(x(T)) + \int_0^T \ell(t, x(t), u(t), b(t), \dot{x}(t), \dot{u}(t), \dot{b}(t)) dt$$

over the controlled sweeping dynamics governed by the so-called play-and-stop operator

$$\left\{ \begin{array}{l} \dot{x}(t) \in -N(x(t); C(t)) + f(x(t), b(t)) \\ \text{for a.e. } t \in [0, T], \quad x(0) = x_0 \in C(0) \subset \mathbb{R}^n \\ \text{with } C(t) = C + u(t), \quad C = \{x \in \mathbb{R}^n \mid \langle a_i, x \rangle \leq 0, i = 1, \dots, m\} \\ \|u(t)\| = 1 \text{ for all } t \in [0, T] \end{array} \right.$$

where the trajectory  $x(t)$  and control  $u(t) = (u_1(t), \dots, u_n(t))$ ,  $b(t) = (b_1(t), \dots, b_n(t))$  functions are absolutely continuous on the fixed interval  $[0, T]$

Observe that we have the intrinsic/hidden state constraints

$$\langle a_i, x(t) - u(t) \rangle \leq 0 \quad \text{for all } t \in [0, T], i = 1, \dots, m$$

due to the construction of the normal cone to  $C(t) = C + u(t)$

**THEOREM** Problem  $(P)$  admits a feasible (absolutely continuous) solution under natural and mild assumptions

## DISCUSSION ON OPTIMAL CONTROL

The formulated optimal control problem for the sweeping process is **not** an optimization problem over a differential inclusion of the type  $\dot{x} \in F(t, x)$ . In our case the **velocity set**  $F(t, x) = -N(x; C(t)) + f(x, b(t))$  is **not fixed** since the sweeping set  $C(t) = C_{u(t)}(t)$  and the perturbation  $f(x, b(t))$  are **different** for each control  $(u, b)$ . Thus we optimize in the **shape** of  $F(t, x)$  which somehow relates this problem to **dynamic shape optimization**. In fact there is **no sense** to formulate any optimization problem for the differential inclusion

$$\dot{x} \in F(t, x) := -N(x; C(t)) + f(x, b(t)), \quad t \in [0, T]$$

when  $C(t)$  is **fixed** since, in major cases, the sweeping inclusion admits a **unique solution** for **every initial point**  $x(0) = x_0 \in C(0)$

## REFORMULATION

Denote  $z := (x, u, b) \in \mathbb{R}^{3n}$ ,  $z(0) := (x_0, u(0), b(0))$

$F(z) := -N(x; C(u)) + f(x, b)$  with  $C(u) := \{x \mid \langle a_i, x \rangle \leq 0, i = 1, \dots, m\}$

Problem (P) can be reformulated as: minimize

$$J[z] = \varphi(z(T)) + \int_0^T \ell(t, z(t), \dot{z}(t)) dt \quad \text{s.t.}$$

$$\dot{z}(t) \in G(z(t)) := F(z(t)) \times \mathbb{R}^n \times \mathbb{R}^n \quad \text{a.e. } t \in [0, T]$$

$$\langle a_i, x(t) - u(t) \rangle \leq 0 \quad \text{for all } t \in [0, T], i = 1, \dots, m$$

$$\|u(t)\| = 1 \quad \text{for all } t \in [0, T]$$

$G(z)$  is unbounded and highly non-Lipschitzian

# DISCRETE APPROXIMATIONS OF SWEEPING TRAJECTORIES

**THEOREM** Fix an arbitrary feasible solution  $\bar{z}(\cdot)$  to (P) and consider discrete partitions

$$\Delta_k := \{0 = t_0^k < t_1^k < \dots < t_k^k = T\} \text{ with } h_k := \max_{0 \leq j \leq k-1} \{t_{j+1}^k - t_j^k\} \downarrow 0$$

Then there is a sequence of piecewise linear functions  $z^k(t) := (x^k(t), u^k(t), b^k(t))$  on  $[0, T]$  with  $\|u_i^k(t_j^k)\| = 1$  for  $i = 1, \dots, m$  satisfying the discretized inclusions

$$x^k(t) = x^k(t_j) + (t - t_j)v_j^k, \quad x(0) = x_0, \quad t_j^k \leq t \leq t_{j+1}^k, \quad j = 0, \dots, k-1$$

with  $v_j^k \in F(z^k(t_j^k))$  on  $\Delta_k$  and such that

$$z^k(t) \rightarrow \bar{z}(t) \text{ uniformly on } [0, T], \quad \int_0^T \|\dot{z}^k(t) - \dot{\bar{z}}(t)\|^2 dt \rightarrow 0$$

The latter implies the a.e. pointwise on  $[0, T]$  convergence of some subsequence of the derivatives  $\dot{z}^k(t) \rightarrow \dot{\bar{z}}(t)$

## DISCRETE CONTROL PROBLEMS

Let  $\bar{z}(\cdot) = (\bar{x}(\cdot), \bar{u}(\cdot), \bar{b}(\cdot))$  be a local optimal solution to  $(P)$ .  
 Consider discrete approximation problems  $(P_k)$  : minimize

$$J_k[z^k] := \varphi(x_k^k) + h_k \sum_{j=0}^{k-1} \ell\left(z_j^k, \frac{z_{j+1}^k - z_j^k}{h_k}\right) \\
 + \sum_{j=0}^{k-1} \int_{t_j^k}^{t_{j+1}^k} \left\| \frac{z_{j+1}^k - z_j^k}{h_k} - \dot{z}(t) \right\|^2 dt$$

over  $z^k := (x_0^k, \dots, x_k^k, u_0^k, \dots, u_k^k, b_0^k, \dots, b_k^k)$  satisfying

$$x_{j+1}^k \in x_j^k + h_k F(x_j^k, u_j^k, b_j^k), \quad j = 0, \dots, k-1, \quad x_0^k = x_0$$

$$\langle a_i, x_j^k - u_j^k \rangle \leq 0, \quad \|u_j^k\| = 1, \quad j = 0, \dots, k-1, \quad i = 1, \dots, m$$



## EXISTENCE OF DISCRETE OPTIMAL SOLUTIONS

**THEOREM** Let  $\varphi$  and  $\ell$  be lower semicontinuous around  $\bar{z}(\cdot)$ . Then each problem  $(P_k)$  admits an optimal solution

**PROOF** employs Attouch theorem on subdifferential convergence for convex extended-real valued functions

## STRONG CONVERGENCE OF DISCRETE APPROXIMATIONS

**THEOREM** Let  $\bar{z}(\cdot) = (\bar{x}(\cdot), \bar{u}(\cdot), \bar{b}(\cdot))$  be a given optimal solution to  $(P)$ . Then any sequence of piecewise linearly extended to  $[0, T]$  optimal solutions  $\bar{z}^k(t)$  of the discrete problems  $(P_k)$  strongly converges to  $\bar{z}(t)$  in the Sobolev space  $W^{1,2}[0, T]$

**PROOF** Using the above result on the strong approximation of trajectories for the sweeping inclusion and relaxation stability

## FURTHER STRATEGY

- For each  $k$  reduce problem  $(P_k)$  to a problem of mathematical programming  $(MP)$  with functional and increasingly many geometric constraints. The latter are given by graphs of the mapping  $F(z) := -N(x; C(u)) + f(x, b)$ , and so  $(MP)$  is intrinsically nonsmooth and nonconvex even for smooth initial data
- Use variational analysis analysis and generalized differentiation (first- and second-order) to derive necessary optimality conditions for  $(MP)$  and then discrete control problems  $(P_k)$
- Explicitly calculate the coderivative of  $F(z)$  entirely in terms of the initial data of  $(P)$
- By passing to the limit as  $k \rightarrow \infty$ , to derive necessary optimality conditions for the sweeping control problem  $(P)$

## NECESSARY OPTIMALITY CONDITIONS FOR (P)

For simplicity consider the case of smooth  $\varphi, \ell$

**THEOREM** Let  $\bar{z}(\cdot)$  be an optimal solution to (P) such that the vectors  $\{a_i\}$  for active constraint  $i \in I(\bar{x}(t) - \bar{u}(t))$  indices are linearly independent. Then there exist a multiplier  $\lambda \geq 0$ , an adjoint arc  $p(t) = (p_x, p_u, p_b)(t)$  absolutely continuous on  $[0, T]$ , and regular Borel measures  $\gamma \in C_+^*([0, T]; \mathbb{R}^m)$  and  $\xi \in C^*([0, T]; \mathbb{R})$  satisfying the following conditions:

the primal-dual relationships for a.e.  $t \in [0, T]$

$$\langle a_i, \bar{x}(t) - \bar{u}(t) \rangle < 0 \implies \eta_i(t) = 0$$

$$\eta_i(t) > 0 \implies \langle a_i, \lambda \nabla_{\dot{x}} \ell(\bar{z}(t), \dot{z}(t)) - q_x(t) \rangle = 0, \quad i = 1, \dots, m$$

where the function  $\eta = (\eta_1, \dots, \eta_m) \in L^\infty([0, T]; \mathbb{R}_+^m)$  is uniquely defined by

$$\dot{\bar{x}}(t) = - \sum_{i=1}^m \eta_i(t) a_i + f(\bar{x}(t), \bar{b}(t))$$

and where  $q(t) = (q_x, q_u, q_b)$  is of bounded variation given by

$$q(t) = p(t) - \int_t^T \left( d\gamma(s), 2\bar{u}(s)d\xi(s) - d\gamma(s), 0 \right), \quad t \in [0, T]$$

$$q_u(t) = \lambda \nabla_{\dot{u}} \ell(t, \bar{z}(t), \dot{\bar{z}}(t)), \quad q_b(t) = \lambda \nabla_{\dot{b}} \ell(t, \bar{z}(t), \dot{\bar{z}}(t)), \quad t \in [0, T]$$

along the **adjoint inclusion**

$$\dot{p}(t) \in \text{co} \left\{ \lambda \nabla_z \ell(\bar{z}(t), \dot{\bar{z}}(t)) + D^* F(\bar{x}(t), \bar{u}(t), \bar{b}(t), -\dot{\bar{x}}(t)) (\lambda \nabla_{\dot{x}} \ell(t) - q_x(t)) \right\}$$

where the **coderivative**  $D^* F$  is calculate via the problem data

Furthermore, we have the **transversality conditions**

$$\begin{aligned} \left( -p_x(T), p_u(T) \right) &\in \left( \lambda \nabla \varphi(\bar{x}(T), 0) + \left( 0, \lambda \nabla_{\dot{u}} \ell(T, \bar{z}(T), \dot{\bar{z}}(T)) \right) \right) \\ &\quad + N(\bar{x}(T) - \bar{u}(T); C) \end{aligned}$$

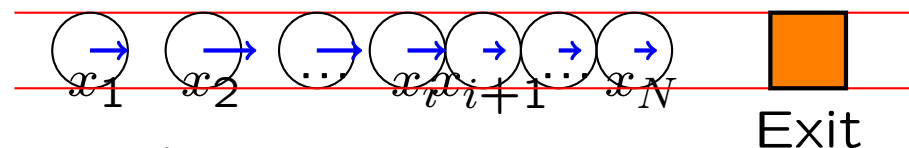
$$p_b(T) = \nabla_b \ell(T, \bar{z}(T), \dot{\bar{z}}(T)) = 0$$

and the **nontriviality condition**

$$\lambda + \|q_u(0)\| + \|p(T)\| \neq 0$$

## CROWD MOTION MODEL

The model is designed to deal with local interactions between individuals in order to describe the dynamics of the pedestrian traffic. This microscopic model for crowd motion rests on the **two principles**: A spontaneous velocity corresponding to the velocity each individual would like to have in the absence of others; the actual velocity is then computed as the **projection** of the spontaneous velocity onto the set of admissible velocities which do not violate a certain **non-overlapping constraint**. We consider  $N(N \geq 2)$  individuals identified to rigid disks with the same radius  $r$  in a corridor (see Fig. 1)



**Fig 1** One-dimensional situation

All the individuals have the same behavior: they want to reach the exist by following the **shortest part**. This problem can be modeled as a sweeping process

$$\begin{cases} -\dot{x}(t) \in N(x(t); C(t)) + f(x(t), b(t)), & x(0) = x_0 \in C(0) \\ C(t) = C + u(t), & u_{i+1}(t) - u_i(t) = 2r, u_1(t) = 0, \|u(t)\| = 1 \\ f(x(t), b(t)) = (s_1 b_1, \dots, s_N b_N) \\ x_{i+1}(T) - x_i(T) > 2r, & i = 1, \dots, N - 1 \end{cases}$$

with **controls in perturbations** and the cost function

$$\text{minimize } J[x, b] = \frac{\|x(T)\|^2}{2} + \int_0^T \frac{\|b(t)\|^2}{2} dt$$

The obtained necessary optimality conditions allow us to determine the **optimal strategy**



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