Revisiting the Lucas Model

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Introduction

seminal paper by Robert E. Lucas (1988)

On the Mechanics of Economic Growth, Journal of Monetary economics

- maximize discounted life time utility
- control variables: consumption, time allocation between education and production
- state variables: physical capital and human capital

Result: **unique solution along a balanced growth path** given

- perfect foresight
- perfect competition in labour and capital markets
- general equilibrium setting

aim of our paper:

analysis of Lucas model in **partial equilibrium setting**

allows two alternative approaches:

1. two-stage optimization framework of Lucas model
   - first stage: maximize lifetime income by choosing division of time between human capital accumulation and production
   - second stage: distribute consumption optimally over lifetime given lifetime income
2. one-state optimization framework of Lucas model
   - one state: fraction of human to physical capital

main result:

in a partial equilibrium setting infinitely many optimal solutions exist for time allocation between work and education

The original Lucas Model

$$\max_{c(\cdot),u(\cdot)} J = \int_0^{\infty} e^{-\rho t} \frac{c(t)^{1-\theta} - 1}{1-\theta} \, \mathrm{d}t, \tag{1}$$

subject to

$$\dot{k}(t) = r(t)k(t) + w(t)h(t)u(t) - c(t), \qquad k(0) = k_0 > 0, \tag{2}$$

$$\dot{h}(t) = h(t)\chi(1 - u(t)), \qquad h(0) = h_0 > 0, \tag{3}$$

$$u(t) \in [0,1], \qquad c(t) \geq 0, \qquad \lim_{t\to\infty} e^{-\int_0^t r(s)\,\mathrm{d}s} k(t) \geq 0, \tag{4}$$

$K(t) = N(t)k(t)$ aggregate physical capital

$L(t) = N(t)h(t)u(t)$ aggregate effective labour

assume stationary population $N(t) = 1 \ \forall t$

$$Y(t) = A\, K(t)^{\alpha} L(t)^{1-\alpha}.$$

$$w(t) = \frac{\partial Y(t)}{\partial L(t)} = A(1-\alpha)\left(\frac{K(t)}{L(t)}\right)^{\alpha},$$

$$R(t) = \frac{\partial Y(t)}{\partial K(t)} = A\alpha\left(\frac{L(t)}{K(t)}\right)^{1-\alpha}.$$

The model considered is a simplification of the original Lucas model:
There is no human capital externality and no population growth

Two-stage optimal control model

partial equilibrium setting: exogenous wage and interest rate

denote $r(t,0) := \int_0^t r(s)\,\mathrm{d}s$

EDUCATION PROBLEM (Problem-U)

$$\max_{u(\cdot)} J_U = \max_{u(\cdot)} \int_0^\infty e^{-r(t,0)} w(t)u(t)h(t)\,\mathrm{d}t, \tag{5}$$

subject to

$$\dot{h}(t) = h(t)\chi(1-u(t)), \qquad h(0) = h_0. \tag{6}$$

CONSUMPTION PROBLEM (Problem-C)

$$\max_{c(\cdot)} J_C = \max_{c(\cdot)} \int_0^\infty e^{-\rho t} \frac{c(t)^{1-\theta} - 1}{1-\theta} \,\mathrm{d}t, \tag{7}$$

subject to

$$\dot{x}(t) = r(t)x(t) - c(t), \qquad x(0) = x_0 + \int_0^\infty e^{-r(t,0)} w(t)u(t)h(t) \,\mathrm{d}t, \tag{8}$$

$$\lim_{t\to\infty} e^{-r(t,0)} x(t) \geq 0. \tag{9}$$

Assumption (A0): The positive-valued function $w(\cdot)$ is such that for any admissible control $u(\cdot)$ and its corresponding trajectory $h(\cdot)$

$$\int_0^{\infty} e^{-r(t,0)} w(t)h(t)u(t)\,\mathrm{d}t < \infty.$$

Proposition

Let (A0) hold. Then the original Lucas problem (1)–(4) is equivalent to the combination of Problem-U and Problem-C.

Proposition

An optimal solution to Problem-C is given by (see, e.g., Acemoglu, 2009)

$$c^*(t) = c_0^* e^{\frac{r(t,0)-\rho t}{\theta}}, \qquad c_0^* = \frac{x(0)}{\int_0^{\infty} e^{-r(t,0)} e^{\frac{r(t,0)-\rho t}{\theta}}\,\mathrm{d}t}. \tag{10}$$

Proposition

Assume (A0) holds. Let $u^*(\cdot)$ be an optimal control in Problem-U and $h^*(\cdot)$ be the trajectory corresponding to it. Then according to Assev and Veliov (2012, 2014) there exists an adjoint variable $\xi(\cdot)$ defined as

$$\xi(t) = \int_t^{\infty} e^{-r(s,0)} w(s) u^*(s) e^{-\chi \int_\tau^s (1-u(\tau))\,\mathrm{d}\tau}\,\mathrm{d}s; \tag{11}$$

that satisfies almost everywhere the adjoint equation

$$-\dot{\xi}(t) = \chi(1-u^*(t))\xi(t) + e^{-r(t,0)} w(t) u^*(t). \tag{12}$$

The optimal control $u^*(\cdot)$ maximizes the Hamiltonian

$$\mathcal{H}(h,u,\xi,t) = e^{-r(t,0)} w(t) h u + \chi \xi h (1-u)$$

evaluated along the optimal path $h = h^*(t)$, $\xi = \xi(t)$ among all admissible $u \in [0,1]$ for almost all $t \in [0,\infty)$. As a consequence,

$$u^*(t) = \begin{cases} 1, & \text{if} \quad e^{-r(t,0)} w(t) - \chi\xi(t) > 0, \\ 0, & \text{if} \quad e^{-r(t,0)} w(t) - \chi\xi(t) < 0, \\ \text{singular}, & \text{if} \quad e^{-r(t,0)} w(t) - \chi\xi(t) = 0. \end{cases} \tag{13}$$

Reducing the State Dimension

partial equilibrium setting: exogenous wage and interest rate

assuming $k(t) > 0$ for all $t \geq 0$, consumption per person can be written as a function of consumption per unit of physical capital $c_f(t) \geq 0$ with $c(t) = c_f(t)k(t)$.

Rewriting the objective function in terms of $c_f(\cdot)$ and substituting in the dynamics of k from equation (2), we obtain the equivalent problem

$$\max_{c_f(\cdot),u(\cdot)} J = \max_{c_f(\cdot),u(\cdot)} \int_0^\infty e^{-\rho t}\left(\ln c_f(t) + \frac{1}{\rho}\left(w(t)u(t)f(t) - c_f(t)\right)\right) \mathrm{d}t, \quad (14)$$

subject to the dynamics of human capital per unit of physical capital $f(t) = h(t)/k(t)$:

$$\dot{f}(t) = f(t)(\chi(1-u(t)) + c_f(t) - r(t)) - f(t)^2 w(t)u(t), \qquad f(0) = \frac{h_0}{k_0} > 0. \quad (15)$$

Theorem

Along a balanced growth path condition

$$\frac{\dot{w}(t)}{w(t)} = r(t) - \chi \quad \text{for all } t \quad \geq 0 \tag{16}$$

is necessary for a singular control to exist over the entire time horizon $[0, \infty)$ *in problem (1) – (4).*

Remark

In case of constant $r(t) \equiv r > 0$ *and* $w(t) \equiv w > 0$ *the condition of the above theorem becomes*

$$r = \chi \quad \text{for all } t \quad \geq 0. \tag{17}$$

Theorem

If condition (17) holds, all controls $u(\cdot)$ *such that*

$$\lim_{t\to\infty} \int_0^t u(s)\,\mathrm{d}s \to \infty \tag{18}$$

are optimal. In particular, all singular controls satisfying (18) are optimal.

General Equilibrium

endogenous wages and interest rates

e.g. Cobb Douglas production function implies the wage rate:

$$w(t) = A(1-\alpha)\left(\frac{k(t)}{u(t)h(t)}\right)^{\alpha}.$$

The first order condition of Education-Problem becomes:

$$e^{-r(t,0)}h(t)^{1-\alpha}A(1-\alpha)^2 k(t)^{\alpha} u(t)^{-\alpha} = \xi(t)\chi.$$

This equation has a unique solution given by

$$u^*(t) = \begin{cases} \left(\frac{\xi(t)\chi e^{r(t,0)}}{h(t)^{1-\alpha}A(1-\alpha)^2 k(t)^{\alpha}}\right)^{\frac{-1}{\alpha}} & \text{if} \quad \left(\frac{\xi(t)\chi e^{r(t,0)}}{h(t)^{1-\alpha}A(1-\alpha)^2 k(t)^{\alpha}}\right)^{\frac{-1}{\alpha}} \leq 1, \\ 1 & \text{if} \quad \left(\frac{\xi(t)\chi e^{r(t,0)}}{h(t)^{1-\alpha}A(1-\alpha)^2 k(t)^{\alpha}}\right)^{\frac{-1}{\alpha}} > 1. \end{cases}$$

Nonlinear Human Capital Production

dynamics of Lucas-Uzawa model for h:

$$\dot{h}(t) = h(t)G(1-u(t)),$$

assume nonlinear function $G(.)$

$$G(x) = \chi x^{\nu} \quad \text{with} \quad \nu < 1$$

The first order condition of the Education-Problem becomes:

$$e^{-r(t,0)}w(t)h(t) - \xi(t)\nu\chi h(t)[1-u(t)]^{\nu-1} = 0.$$

Then the boundary solution $u(t) = 1$ will never be optimal because $e^{-r(t,0)}w(t)h(t)$ is strictly positive for all t. The assumption $\nu < 1$ implies that $u^*(t)$ is uniquely determined by

$$u^*(t) = \begin{cases} 0, & \text{if} \quad 1 - \left(\frac{e^{-r(t,0)}w(t)h(t)}{\xi(t)\nu\chi h(t)}\right)^{\frac{1}{\nu-1}} < 0 \\ 1 - \left(\frac{e^{-r(t,0)}w(t)h(t)}{\xi(t)\nu\chi h(t)}\right)^{\frac{1}{\nu-1}} & \text{if} \quad 1 - \left(\frac{e^{-r(t,0)}w(t)h(t)}{\xi(t)\nu\chi h(t)}\right)^{\frac{1}{\nu-1}} \geq 0. \end{cases}$$

- we have reformulated the Lucas model in a partial equilibrium setting
- for the partial equilbrium setting we indicate alternative solution methods
  1. two stage optimization
  2. one state optimization
- in the partial equilibrium setting multiple optimal solutions may exist
  switching between education and work may be optimal in such a setting
- to select a unique optimal solution we provide two mechanisms:
  1. general equilibrium setting
  2. nonlinear human capital accumulation